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HYPERELASTIC POLYNOMIAL MODELS IN PLASTICS BEHAVIOUR ANALYSES

Piotr Żach

Warsaw University of Technology
Department of Materials Engineering
Narbutta Street 84, 02-524 Warsaw, Poland
tel.: +48 22 2348620, fax: +48 22 2348622
e-mail: pzach@simr.pw.edu.pl

Abstract

The issue of vibration damping occurs in many mechanical problems in the operation of various devices, in the automotive industry engineering, aerospace. In the case where damping by selection of the masses and dimensions is not possible due to various reasons or for other reasons it was abandoned, vibration dampers are used. Minimization of adverse impact of dynamic interaction effects is an important research and technical problem. Passive energy absorbers used today (complex of bumpers, passenger lifts buffers typically allows to the safe dissipation of energy within a certain range of loads. In the case of high-impact loading variability is desirable to use an adaptive energy absorption system capable of rapid change their dynamic characteristics. The main issue in the analysis of interactions impact of dynamic loads on objects is dispersion (dissipation) of kinetic energy during impact. In constructions of polymer composite structures the sources of energy dissipation are: matrix of polystructural viscoelasticity, morphology of material, defect of structure, thermoplastic and viscoplastic dumping. In construction of machines more and more often be practical used materials name hyper-elastic. To description of elastic proprieties it's possible to use the worked out theories well in which the most important they are the multinomial models and them special coincidence: Mooney's – Rivlin's, Yeoh's. In this paper, is presented numerical behaviour analyses materials with large deformation.

Keywords: *incompressible materials, elastomers, oscillation, very large reversible deformations, vibration*

1. Introduction

Hyperelastic materials, particularly elastomers, are often used in vibrating systems – interesting examples are the shimmy vibration damper, impact limiters, bumpers. The issue of vibration damping occurs in many mechanical problems in the operation of various devices [1], in the automotive industry engineering, aerospace. In the case where damping by selection of the masses and dimensions is not possible due to various reasons or for other reasons it was abandoned, vibration dampers are used [2].

Minimization of adverse impact of dynamic interaction effects is an important research and technical problem. Passive energy absorbers used today (complex of bumpers, passenger lifts buffers typically allows to the safe dissipation of energy within a certain range of loads [1, 3]. In the case of high-impact loading variability is desirable to use an adaptive energy absorption system capable of rapid change their dynamic characteristics [4]. The main issue in the analysis of interactions impact of dynamic loads on objects is dispersion (dissipation) of kinetic energy during impact [5].

The energy dissipations the irreversible process of transforming structured forms macroscopic movement of energy distributed randomly in a number of degrees of freedom, usually in the thermal energy of the movement of microparticles (increasing the relative amount of thermal energy of the system and increases its entropy) [6]. Dissipative processes occur in systems in which the interaction between mating components, or within the structure of the material occur as a result of friction, viscous characteristics of the material, among others following: changes in process temperature, deformation speed.

Hyperelastic material models are used to analyse the spatial arrangements in the states stresses – guiding the vibration equations for the direction of motion, need to receive the simplifying assumptions. In the functional which describes deformation energy are two separate parts: the figural – assuming incompressibility of the structure in accordance with the condition (1), and volume – you should consider the relationship between deformations specific Poisson's ratio

$$\lambda_1 \lambda_2 \lambda_3 = 1\,. \tag{1}$$

For substances with isotropic properties are assumed to deformation under (2):

$$\begin{aligned} \varepsilon_2 &= -\nu\,\varepsilon_1 = -\nu(\lambda_1 - 1) = -\nu\,\lambda_1 + \nu, \\ \varepsilon_3 &= -\nu\,\varepsilon_1 = -\nu(\lambda_1 - 1) = -\nu\,\lambda_1 + \nu. \end{aligned} \tag{2}$$

After entering these simplifications to the appropriate due to the application of the model, complete differentiation according to the relation (3):

$$\sigma_1 = \frac{\partial W}{\partial \lambda_1}\,. \tag{3}$$

To obtain the equation of motion for the main deformations described dimensionless variable we substitute equation (4):

$$\lambda_1 = \frac{x_i + l_i}{l_i}\,, \tag{4}$$

where:
x_i – change in length caused by a given load,
l_i – initial length,
i – takes the values: 1, 2, 3,
resulting from the equation (5):

$$\lambda_1 = \varepsilon_1 + 1\,, \tag{5}$$

which consequently leads to the equation with a dimensionless coordinate *x*.

2. Polynomial model and its special cases

In the model of polynomial written by the equation (6):

$$W = \sum_{i+j=0}^{N} C_{ij} (\bar{I}_1 - 3)^i (\bar{I}_2 - 3)^j\,, \tag{6}$$

where:
W – figural deformation energy,
$\bar{I}_1$ – first invariant strain tensor Cauchy's–Green's, defined as (7),
$\bar{I}_2$ – second invariant strain tensor Cauchy's–Green's, written as (8),
C_{ij} – material constants,
C_{00} – material constant, $C_{00} = 0$,
μ_0 – initial transverse modulus of elasticity initial ratio,

$$\bar{I}_1 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2\,, \tag{7}$$

$$\bar{I}_2 = (\bar{\lambda}_1)^{-2} + (\bar{\lambda}_2)^{-2} + (\bar{\lambda}_3)^{-2}\,, \tag{8}$$

after the substitution of dependences (9):

$$\lambda_2^2 = \frac{1}{\lambda_1},\ \lambda_3^2 = \frac{1}{\lambda_1}, \tag{9}$$

will take the form (10):

$$\lambda_1 \lambda_2 \lambda_3 = 1\,. \tag{10}$$

Consider the special cases of equation (6), namely: reduced polynomial model – Yeoh's and two-parametric Mooney's – Rivlin's. Yeoh's model is used for hyperelastic linear materials, nearly incompressible. Model is based on R. S. Rivlin's observations and studies [7]. It was assumed that the material can be described by the density function of figural deformation, which is a sequence of power series invariants $\bar{I}_1, \bar{I}_2, \bar{I}_3$. For compressible material model depends only on the first invariant $\bar{I}_1$. The model is called as reduced polynomial model. The original model proposed by Yeoh's [8, 9] had the form (11):

$$W = \sum_{i=0}^{N} C_i (\bar{I}_1 - 3)^i\,. \tag{11}$$

Yeoh's model in the general form is defined as (12):

$$W = \sum_{i=0}^{N} C_i (\bar{I}_1 - 3)^i + \sum_{k=1}^{N} \frac{1}{D_k} C_i (J_{el} - 1)^{2k}\,, \tag{12}$$

taking into account (1) and (9) the equation will take the form (13):

$$\begin{aligned} W = {} & C_1\left(-3 + \frac{2}{\lambda_1} + \lambda_1^2\right) + C_2\left(-3 + \frac{2}{\lambda_1} + \lambda_1^2\right)^2 + C_3\left(-3 + \frac{2}{\lambda_1} + \lambda_1^2\right)^3 + C_4\left(-3 + \frac{2}{\lambda_1} + \lambda_1^2\right)^4 + \\ & + C_5\left(-3 + \frac{2}{\lambda_1} + \lambda_1^2\right)^5 + C_6\left(-3 + \frac{2}{\lambda_1} + \lambda_1^2\right)^6 + \frac{(\lambda_1\lambda_2\lambda_3 - 1)^2}{D_1} + \frac{(\lambda_1\lambda_2\lambda_3 - 1)^4}{D_2} + \\ & + \frac{(\lambda_1\lambda_2\lambda_3 - 1)^6}{D_3} + + \frac{(\lambda_1\lambda_2\lambda_3 - 1)^8}{D_4} + \frac{(\lambda_1\lambda_2\lambda_3 - 1)^{10}}{D_5} + \frac{(\lambda_1\lambda_2\lambda_3 - 1)^{12}}{D_6}. \end{aligned} \tag{13}$$

After taking into account (4) and (5) the figural deformation will be write as (14):

$$\begin{aligned} \sigma_1 = {} & 2\left(\frac{l+x}{l^2} - \frac{l^3}{(l+x)^2}\right)C_1 + \frac{x^2(3l+x)(2l^8(l+x)^4}{l^{10}(l+x)^5} C_2 + \\ & + l^2x^2(l+x)(3l+x)(l^2(l+x)(3l^2(l+x)C_3 + 4x^2(3l+x)C_4) + 5x^4(3l+x)^2 C_5) + 6x^8(3l+x)^4 C_6)). \end{aligned} \tag{14}$$

Is appropriate to adopt other conditions, for the part describing the volumetric strain. It is necessary to take into account of developments in the structure of the material and resulting from changes in temperature as a result of segmental motion of molecules [10].

To describe the state of stress and elastic deformation of materials under the influence of temperature change concepts and the equation used in the theory of thermoelasticity, on which work began Duhamel [11, 12].

It is assumed that the body is free from stresses at reference temperature. The current absolute temperature was determined and the temperature difference is equal to $\theta \equiv T - T_0$. It is assumed that in the considered range of temperature parameters, characteristic of the material, such as modulus of elasticity *E,* Poisson's number ν and the coefficient of thermal expansion are not changed. With these assumptions the relative lengthening of the material can be calculated as the sum of elongation resulting from the forces and thermal deformations. In the state of elastic deformation of the constituent component ε_x, ε_y, ε_z with the components of the stress σ_x, σ_y, σ_z Hooke's law defines (15).

$$\begin{aligned}
\varepsilon_x^s &= \frac{1}{E}[\sigma_x + \nu(\sigma_y + \sigma_z)],\\
\varepsilon_y^s &= \frac{1}{E}[\sigma_y + \nu(\sigma_z + \sigma_x)],\\
\varepsilon_z^s &= \frac{1}{E}[\sigma_z + \nu(\sigma_x + \sigma_z)].
\end{aligned} \tag{15}$$

The components of the elastic deformation ε_x^s, ε_y^s, ε_z^s with the components of the stress $\sigma_x = 0$, $\sigma_y = 0$, $\sigma_z = 0$.

Taking into account the dependence (15), (16):

$$\begin{aligned}
\Delta T_1 &= T_1 - T_0,\\
\Delta T_2 &= T_2 - T_1,\\
\Delta T_3 &= T_3 - T_2
\end{aligned} \tag{16}$$

and receiving simplification in accordance with the (17):

$$\begin{aligned}
\alpha &= \alpha_1 = \alpha_2 = \alpha_3,\\
\Delta T &= \Delta T_1 = \Delta T_2 = \Delta T_3,
\end{aligned} \tag{17}$$

get a description of the Yeoh's material model (12) with the separation of figural and volume (18):

$$\begin{aligned}
\sigma_1 = {} & 2\left(\frac{l+x}{l^2} - \frac{l^3}{(l+x)^2}\right)C_1 + \frac{x^2(3l+x)(2l^8(l+x)^4}{l^{10}(l+x)^5}C_2 +\\
& + l^2x^2(l+x)(3l+x)(l^2(l+x)(3l^2(l+x)C_3 + 4x^2(3l+x)C_4) + 5x^4(3l+x)^2C_5) + 6x^8(3l+x)^4C_6)) +\\
& + \frac{6(l+x)^2\left(\frac{(l+x)^3}{3l^3(\alpha(T_1-T_0)+1))^3} - 1\right)}{l^3D_1(\alpha(T_1-T_0)+1))^3} + \frac{12(l+x)^2\left(\frac{(l+x)^3}{3l^3(\alpha(T_1-T_0)+1))^3} - 1\right)^3}{l^3D_2(\alpha(T_1-T_0)+1))^3} +\\
& + \frac{18(l+x)^2\left(\frac{(l+x)^3}{3l^3(\alpha(T_1-T_0)+1))^3} - 1\right)^5}{l^3D_3(\alpha(T_1-T_0)+1))^3} + \frac{24(l+x)^2\left(\frac{(l+x)^3}{3l^3(\alpha(T_1-T_0)+1))^3} - 1\right)^7}{l^3D_4(\alpha(T_1-T_0)+1))^3} +\\
& + \frac{30(l+x)^2\left(\frac{(l+x)^3}{3l^3(\alpha(T_1-T_0)+1))^3} - 1\right)^9}{l^3D_5(\alpha(T_1-T_0)+1))^3} + \frac{36(l+x)^2\left(\frac{(l+x)^3}{3l^3(\alpha(T_1-T_0)+1))^3} - 1\right)^{11}}{l^3D_6(\alpha(T_1-T_0)+1))^3}.
\end{aligned} \tag{18}$$

where:

α – linear expansion coefficient of the material,
T_1 – temperature of the material produced in the structure as a result of external mechanical influences, e.g. compression,
T_0 – initial temperature.

Model of Mooney – Rivlin is correct for the non-compressible material, for which deformation are in the range 100% to 200% (among others cross-linked natural rubber elastomers, e.g. rubber used for the production of tires). Model of Mooneya – Rivlin occur in two-, three-, five- or nine-parameters variations. It is the most widely used model based on the invariant form of strain energy density functional. It is a modification of the Mooney's model involving determinations the change of material constants. The basic model has been updated with the last part takes into account the slight compressibility. The most common, providing a good representation of the behaviour of the elastomer in the range of 100% strain for the stretching and 30% compression, a variety of two-parameter which is described by the equation (19):

$$W = C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + \frac{1}{D}(J_{el} - 1)^2, \tag{19}$$

where:
C_{10}, C_{01} – material constants, C10 temperature dependent,
μ_0 – initial modulus ratio, defined as the (20):

$$\mu_0 = 2(C_{10} + C_{01}). \tag{20}$$

After taking into account (1), (5) and (9) we obtain the equation (21) and substituting (3) and (4) we obtain (22):

$$W = C_{01}\left(\frac{1}{\lambda_1^2} + 2\lambda_1 - 3\right) + C_{10}\left(\frac{2}{\lambda_1} + \lambda_1^2 - 3\right), \tag{21}$$

$$\sigma_1 = 2C_{10}\left(\frac{l+x}{l^2} - \frac{l}{(l+x)^2}\right) + C_{01}\left(\frac{2}{l} + \frac{2l^2}{(l+x)^3}\right). \tag{22}$$

After taking into account (15), (16) and completion of the measures, the model Mooney's – Rivlin's equation for the figural deformation energy will take the form (23):

$$W = \left(\frac{l^2}{(l+x)^2} + \frac{2(l+x)}{l} - 3\right)C_{01} + \left(\frac{2l}{l+x} + \frac{(l+x)^2}{l^2} - 3\right)C_{10} + \\ + \frac{2}{D}\left(\frac{(l+x_1)(l+x_2)(l+x_3)}{l^3((1+(T_1-T_0)\alpha_1)^3 + (1+(T_2-T_1)\alpha_2)^3 + (1+(T_3-T_2)\alpha_3)^3} - 1\right). \tag{23}$$

Based on the analyses it can be concluded that in the incompressible elastomers, the consequence of non-linearity present in the denominator is the total displacement of the second component.

Considering the case of the behaviour of the material in the compression test, taking into account the effect of temperature, in clear form, substituting (3) and taking (17), write the (24):

$$\sigma_1 = 2C_{10}\left(\frac{l+x}{l^2} - \frac{l}{(l+x)^2}\right) + C_{01}\left(\frac{2}{l} + \frac{2l^2}{(l+x)^3}\right) + \frac{4(l+x)^2\left(\frac{(l+x)^3}{3l^3(\alpha(T_1-T_0)+1)^3} - 1\right)}{Dl^3(\alpha(T_1-T_0)+1)^3}. \tag{24}$$

Considering, a special case of incompressible material, considered in the analysis of phenomena, by I. M. Word [13], as defined for the polynomial model in equation (6), adopted into consideration the development of the words (level model $n = 1$) in accordance with (25):

$$W = C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3). \tag{25}$$

After taking into account (1), (4), (5), we obtain the equation of motion describes dependence (26):

$$m\ddot{x} + 2k\,l^{-2}\left(l + x - \frac{l^3}{(l+x)^2}\right) = F(t). \tag{26}$$

where:
k – elastic modulus.

3. Conclusion

In the analysed models structural description of the material has been built on the assumption of non-linear behaviour of polymers. It has been shown that it is appropriate to consider all the factors determining the work material, i.e. take into account the description of the property changes resulting from phase transformations resulting from e.g. temperatures. Based on the analyses it can be concluded that in the incompressible elastomers, the consequence of non-linearity occurring in denominator is the total displacement of the second component.

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