DIFFERENTIAL AND RECURRENCE UNIFIED REYNOLDS EQUATIONS AND MEGA ALGORITHM FOR THEIR NUMERICAL SOLUTIONS

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Abstract

The objective of the research under the paper topic is an analytical, unified formulation of a new standardized view of general solution of hydrodynamic problem using algorithm to determine changes of the components of the velocity vector, the distributions of hydrodynamic pressure, load carrying capacity, of slide bearings with cooperating curvilinear, orthogonal surfaces that are lubricated with a various non-Newtonian lubricants. In this paper for non-Newtonian lubricants are questioning the hitherto prevailing assumptions using in hydrodynamic theory of lubrication such as constant value of lubricant viscosity and pressure in the thickness of lubricating gap i.e. in gap height direction.

Finally, the non-homogeneous partial differential equation generated with variable coefficients that is the result of the various boundary conditions being imposed that are different for each problem solved is an equation that determines the distributions of hydrodynamic pressure values. This equation is to be written in the form of a unified non-homogenous partial recurrence equation with variable coefficients. The Authors foresee that a mega-algorithm will be developed for the solution of this equation in a numerical form. This equation in particular cases is an equivalent of modified Reynolds equations in the research that has been conducted so far concerning the hydrodynamic theory of lubrication.

Keywords: *partial differential form, partial recurrence form, unified Reynolds equation, mega algorithm*

1. Introduction

The authors accepts a hypothetical assumption that investigations of the hydrodynamic lubrication of slide bearings starting from the foundations of the problem shall result in questioning in some areas of solutions the basic simplifications that have been in use so far, e.g. the constant value of the viscosity of the lubricating liquid and of the hydrodynamic pressure on the thickness of the lubricating gap, lack of interaction of material coefficients of the superficial layer of lubricated surfaces on the viscosity of the lubricating agent. The theory of hydrodynamic lubrication that has been valid so far is based on the abovementioned simplification assumptions and it leads to Reynolds equations that are more or less modified and that determine the distributions of the values of the hydrodynamic pressure [5]. The research practice that has been accepted so far by many authors for the formulation of various problems in the area of hydrodynamics comes down to modifications of the Reynolds equation that was derived 100 years ago without any thorough derivations; one forgets that it is the Reynolds

equation that is the result of an imposing concrete boundary conditions that are different in almost each problem on the components of the distribution of the velocities of the lubricating liquid in the bearing gap. Furthermore, no attention is paid to the curvature of lubricated surfaces and, practically speaking, the same equation is accepted in cylindrical, rectangular or spherical coordinates. The authors assume that the abovementioned simplifications may lead to numerous incompatibilities of the results of numerical and empirical research [4].

2. The system of partial differential equations

We show following system of non-linear basic partial differential equations describing the lubrication of two curvilinear non-rotational surfaces [1-3]:

Equation of continuity:

$$\frac{\partial}{\partial t}(\rho h_1 h_3)+\frac{\partial}{\partial \alpha_1}(\rho v_1 h_3)+\frac{\partial}{\partial \alpha_2}(\rho v_2 h_1 h_3)+\frac{\partial}{\partial \alpha_3}(\rho v_3 h_1)=0, \tag{1.1}$$

Equation of motion:

$$X_i(v_1,v_3)=-\frac{1}{h_i}\frac{\partial p}{\partial \alpha_i}+\frac{\partial}{\partial \alpha_2}\left[\eta_p(v_1,v_3,\alpha,\beta)\frac{\partial v_i}{\partial \alpha_2}\right], \tag{1.2}$$

$$(\alpha+2\beta)\frac{\partial}{\partial \alpha_2}\left[\left(\frac{\partial v_1}{\partial \alpha_2}\right)^2+\left(\frac{\partial v_3}{\partial \alpha_2}\right)^2\right]=\frac{\partial p}{\partial \alpha_2}, \tag{1.3}$$

where for i=1,3., we have:

$$X_i(v_1,v_3)\equiv\rho\left(\frac{\partial v_i}{\partial t}+\frac{v_1}{h_1}\frac{\partial v_i}{\partial \alpha_1}+v_2\frac{\partial v_i}{\partial \alpha_2}+\frac{v_3}{h_3}\frac{\partial v_i}{\partial \alpha_3}+\frac{v_1 v_3}{h_1 h_3}\frac{\partial h_i}{\partial \alpha_{4-i}}-\frac{v_{4-i}^2}{h_1 h_3}\frac{\partial h_{4-i}}{\partial \alpha_i}\right). \tag{1.4}$$

Dimensional apparent viscosity $\eta_p\equiv\eta_p(v_1,v_3,\alpha,\beta)$ is obtained from Rivlin-Ericksen dependencies. We denote: t – time, ρ – fluid density, δ_{j0} – Kronecker symbol. The unknown functions are: velocity components v_1, v_2, v_3, pressure p. Velocity components v_i, pressure p, apparent viscosity function η_p, are presented in following series expansions in relation to small parameter [6, 7, 8]:

$$v_i(\alpha_1,\alpha_2,\alpha_3)=v_{i0}(\alpha_1,\alpha_2,\alpha_3)+Dv_{i1}(\alpha_1,\alpha_2,\alpha_3)+\ldots+D^j v_{ij}(\alpha_1,\alpha_2,\alpha_3)+\ldots\ , \tag{2.1}$$

$$p(\alpha_1,\alpha_2,\alpha_3)=p_{d0}(\alpha_1,\alpha_3)+Dp_{d1}(\alpha_1,\alpha_2,\alpha_3)+\ldots+D^j p_{dj}(\alpha_1,\alpha_2,\alpha_3)+\ldots\ , \tag{2.2}$$

$$\eta_p(v_1,v_3,\alpha,\beta)=\eta_0\left[1+D\eta_{p1}(\alpha_1,\alpha_2,\alpha_3)+\ldots+D^j\eta_{pj}(\alpha_1,\alpha_2,\alpha_3)+\ldots\right], \tag{2.3}$$

$$\eta_{pj}(v_1,v_3)=\frac{1}{j!}\left\{\frac{\partial^j\eta_p(v_1,v_3,D)}{\partial D^j}\right\}_{D=0}\quad for\ j=1,2,3,\ldots\ . \tag{2.4.1}$$

$$D=D_\alpha\equiv\frac{\alpha\omega}{\eta_0}\quad or\quad D=D_\beta\equiv\frac{\beta\omega}{\eta_0}. \tag{2.4.2}$$

where i=1,2,3; j=0,1,2,…, ω – linear velocity of cooperating surface, D – Deborah number. We denote: η_0 – characteristic dimensional value of classical dynamic viscosity, η_{pj} dimensionless expansion coefficients whereas for j=0 we have $\eta_{p0}\equiv1$ and $\eta_{pj}\equiv\eta_{pj}(v_1,v_3)$ for j=1,2,.... Moreover α, β – first and second pseudoviscosity coefficient in Pas^2, η_0 – characteristic constant dynamic viscosity value in Pas, $\mathbf{A}_1$ – velocity deformation tensor in s^{-1} [1].

Now we put series (2.1-2.3) into the system of partial differential equations (1.1)-(1.3). Multiplying the series by Cauchy method, equating the coefficients of the like powers of small parameter D, we obtain a sequence of following systems of non-linear (for $X_{ij}\neq 0$), or linear (for $X_{ij}=0$) partial equations [9]:

$$X_{ij}(v_{i0}, v_{i1}, .., v_{ij}) + \frac{1}{h_i}\frac{\partial p_{dj}}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_2}\left(\eta_0 \frac{\partial v_{ij}}{\partial \alpha_2}\right) + \frac{\partial}{\partial \alpha_2}\left(\eta_0 S_{ij}\right), \quad i = 1,3, \tag{3}$$

$$\frac{\partial p_{dj}}{\partial \alpha_2} = \zeta \frac{\partial F_j}{\partial \alpha_2}, \tag{4}$$

$$\delta_{j0}\frac{\partial}{\partial t}\left(\rho h_1 h_3\right) + \frac{\partial}{\partial \alpha_1}\left(\rho v_{1j} h_3\right) + \frac{\partial}{\partial \alpha_2}\left(\rho v_{2j} h_1 h_3\right) + \frac{\partial}{\partial \alpha_3}\left(\rho v_{3j} h_1\right) = 0, \tag{5}$$

for i=1,3; j=0,1,2,…, $\zeta \equiv (1+2\beta/\alpha)\eta_0/\omega$ where:

$$S_{i0} \equiv 0, \quad S_{i1} = \frac{\partial v_{i0}}{\partial \alpha_2}\eta_{p1}, \quad S_{i2} = \frac{\partial v_{i0}}{\partial \alpha_2}\left(\eta_{p2}\right) + \frac{\partial v_{i1}}{\partial \alpha_2}\eta_{p1}, ..., \quad S_{ij} = S_{ij}\left(v_{1j-1}, v_{3j-1}\right),$$

$$F_0 = 0, \quad F_1 = G_0, \quad F_2 = 2G_{11}, \quad F_3 = 2G_{12} + G_{13}, ..., \quad F_j = F_j\left(v_{1j-1}, v_{3j-1}\right),$$

$$\eta_{p1} = \eta_{p1}\left(v_{10}, v_{30}\right), \eta_{p2} = \eta_{p2}\left(v_{11}, v_{31}\right), ..., \quad \eta_{pj} = \eta_{pj}\left(v_{1j-1}, v_{3j-1}\right), \tag{6}$$

$$G_0 \equiv \left(\frac{\partial v_{10}}{\partial \alpha_2}\right)^2 + \left(\frac{\partial v_{30}}{\partial \alpha_2}\right)^2, \quad G_{11} = \frac{\partial v_{10}}{\partial \alpha_2}\frac{\partial v_{11}}{\partial \alpha_2} + \frac{\partial v_{30}}{\partial \alpha_2}\frac{\partial v_{31}}{\partial \alpha_2},$$

$$G_{12} = \frac{\partial v_{10}}{\partial \alpha_2}\frac{\partial v_{12}}{\partial \alpha_2} + \frac{\partial v_{30}}{\partial \alpha_2}\frac{\partial v_{32}}{\partial \alpha_2}, \quad G_{13} = \left(\frac{\partial v_{11}}{\partial \alpha_2}\right)^2 + \left(\frac{\partial v_{31}}{\partial \alpha_2}\right)^2, ...$$

System of Eqs.(3)-(5) for $X_{ij}=0$, determines following unknown functions: v_{1j}, v_{2j}, v_{3j}, p_{dj}, for i=1,3; j=0,1,2,... where X_{ij}^* – inertia force, and convection transport obtained after Pickard approximation procedures:

$$X_{ij}^{(k)} \xrightarrow{k\to\infty} X_{ij}^*. \tag{7}$$

Symbols $h_i(\alpha_3)$ for i=1,3 denote Lame coefficients for rotational surfaces and its non-monotone generating lines. For non-rotational surfaces, we have: $h_i(\alpha_1,\alpha_3)$ for i=1,3.

3. Boundary conditions

Since the two cooperating surfaces are moving, and there can be slip, hence the boundary conditions (for i=1,2,3; j=0,1,2,…) have the following form [6]:

$$v_{ij}(\alpha_1, \alpha_2 = 0, \alpha_3, t) = \delta_{j0} U_i(\alpha_1, \alpha_3, t), \tag{8a}$$

$$v_{ij}(\alpha_1, \alpha_2 = h, \alpha_3, t) = \delta_{j0} U_{ip}(\alpha_1, \alpha_3, t), \tag{8b}$$

where δ_{j0} denotes Delta Kronecker Symbol, h denotes gap height. Functions $U_i \geq 0$, $U_{ip} \geq 0$ can be continuous, constant or variable but not arbitrary in general.

4. Solutions of system differential equations

Integrating twice equations (3) solutions with respect to variable α_2 under conditions (8a), (8b), then if functions X and Z are uniform convergent to X^*, Z^* after Pickard procedure (6), hence we obtain [2, 6]:

$$v_{ij} = \frac{1}{h_i}\frac{\partial p_{dj}}{\partial \alpha_i} A_\eta + \int_0^{\alpha_2}\left[\frac{1}{\eta_0}\int_0^{\alpha_2} X_{ij}^*\, d\alpha_2 - S_{ij}\right] d\alpha_2 - A_s \int_0^{h}\left[\frac{1}{\eta_0}\int_0^{\alpha_2} X_{ij}^*\, d\alpha_2 - S_{ij}\right] d\alpha_2 +$$

$$+ \delta_{j0}\left[A_s\left(U_{ip} - U_i\right) + U_i\right], \quad for \quad i = 1,3; j = 0,1,2..., \tag{9}$$

where:

$$A_s(\alpha_1,\alpha_2,\alpha_3) \equiv \int_0^{\alpha_2}\frac{1}{\eta_0} d\alpha_2 \left(\int_0^{h}\frac{1}{\eta_0} d\alpha_2\right)^{-1}, \quad A_\eta(\alpha_1,\alpha_2,\alpha_3) \equiv \int_0^{\alpha_2}\frac{\alpha_2}{\eta_0} d\alpha_2 - A_s(\alpha_1,\alpha_2,\alpha_3)\int_0^{h}\frac{\alpha_2}{\eta_0} d\alpha_2. \tag{10}$$

We integrate continuity equation (5) with the respect to variable α_2, i.e. in gap height direction. Imposing the condition (8a) for i=2 i.e. $v_{2j}=U_2\delta_{j0}$ for $\alpha_2=0$, upon velocity component v_{2j} in gap height direction α_2, we get the following solution [5]:

$$v_{2j} = \frac{\rho_0}{\rho} U_2 \delta_{j0} - \frac{1}{\rho h_1 h_3}\left[\int_0^{\alpha_2}\delta_{j0}\frac{\partial}{\partial t}(\rho h_1 h_3) d\alpha_2 + \int_0^{\alpha_2}\frac{\partial}{\partial \alpha_1}\left(\rho v_{1j}\, h_3\right) d\alpha_2 + \int_0^{\alpha_2}\frac{\partial}{\partial \alpha_3}\left(\rho v_{3j} h_1\right) d\alpha_2\right], \tag{11}$$

where j=0,1,2,… and $\rho_0 \equiv \rho(\alpha_1, \alpha_2 = 0, \alpha_3, t)$.

Imposing the condition (8b) for i=2 i.e. $v_{2j}=U_{2p}\delta_{j0}$ for $\alpha_2=h$, upon velocity component (11) in gap height direction α_2, we get the following expression:

$$h_1 h_3\left(\rho_0 U_2 - \rho_h U_{2p}\right)\delta_{j0} = \int_0^{h}\delta_{j0}\frac{\partial}{\partial t}(\rho h_1 h_3) d\alpha_2 + \int_0^{h}\frac{\partial}{\partial \alpha_1}\left(\rho v_{1j} h_3\right) d\alpha_2 + \int_0^{h}\frac{\partial}{\partial \alpha_3}\left(\rho v_{3j} h_1\right) d\alpha_2, \tag{12}$$

where j=0,1,2,… and $\rho_h \equiv \rho(\alpha_1, \alpha_2 = h, \alpha_3, t)$.

Differentiating the definite integrals with variable limits of integration, we obtain the following formulae [2, 4, 5]:

$$\int_0^{h}\frac{\partial}{\partial \alpha_i}\left(\frac{\rho g}{h_i} v_{ij}\right) d\alpha_2 = \frac{\partial}{\partial \alpha_i}\int_0^{h}\left(\frac{\rho g}{h_i} v_{ij}\right) d\alpha_2 - \frac{\partial h}{\partial \alpha_i}\frac{\rho_h g}{h_i} v_{ij}(\alpha_1, \alpha_2 = h, \alpha_3), \tag{13}$$

for i=1,3; j=0,1,2,… .

Because $h_2=1$, hence:

$$g = h_1 h_2 h_3 = h_1 h_3. \tag{14}$$

We put identity (13) in expression (12) and we take into account boundary conditions (8a), (8b) i.e.:

$$\begin{aligned} &v_{10}(\alpha_1, \alpha_2 = h, \alpha_3) = U_2, \quad v_{1j}(\alpha_1, \alpha_2 = h, \alpha_3) = 0 \quad for \quad j = 1,2,3,..., \\ &v_{30}(\alpha_1, \alpha_2 = h, \alpha_3) = U_{2p}, \quad v_{3j}(\alpha_1, \alpha_2 = h, \alpha_3) = 0 \quad for \quad j = 1,2,3,.... \end{aligned} \tag{15}$$

Hence for j=0,1,2,… we obtain [6]:

$$h_1 h_3 \left(\rho_0 U_2 - \rho_h U_{2p}\right)\delta_{j0} + \left(\frac{\partial h}{\partial \alpha_1} h_3 U_{1p} + \frac{\partial h}{\partial \alpha_3} h_1 U_{3p}\right)\rho_h \delta_{j0} =$$

$$= \frac{\partial}{\partial \alpha_1}\int_0^h \left(\rho\, v_{1j}\, h_3\right) d\alpha_2 + \frac{\partial}{\partial \alpha_3}\int_0^h \left(\rho\, v_{3j}\, h_1\right) d\alpha_2 + h_1 h_3 \left(\frac{\partial}{\partial t}\int_0^h \rho\, d\alpha_2 - \rho_h \frac{\partial h}{\partial t}\right)\delta_{j0}. \qquad (16)$$

From formula (16) unknown pressure functions p_j can be calculated as functions of velocity components v_{1j}, v_{3j} for j=0,1,2,… . Fluid velocity vector components v_{1j}, v_{3j} presented by formula (9) we put in Eq.(16). Thus, we obtain [6]:

$$\frac{1}{h_1 h_3}\frac{\partial}{\partial \alpha_1}\left[\frac{h_3}{h_1} A_\eta^*(\alpha_1,\alpha_3)\frac{\partial p_j}{\partial \alpha_1}\right] + \frac{1}{h_1 h_3}\frac{\partial}{\partial \alpha_3}\left[\frac{h_1}{h_3} A_\eta^*(\alpha_1,\alpha_3)\frac{\partial p_j}{\partial \alpha_3}\right] = A_{\Sigma j}(\alpha_1,\alpha_3,t), \qquad (17)$$

for j=0,1,2,…; $h_1=h_1(\alpha_1,\alpha_3)$, $h_3=h_3(\alpha_1,\alpha_3)$, $p_{dj}(\alpha_1,\alpha_2=0,\alpha_3)$, and

$$A_{\Sigma j} = -\frac{1}{h_1 h_3}\sum_{i=1,3}\frac{\partial}{\partial \alpha_i}\left\{\frac{g}{h_i}\left[\int_0^h\left(\rho\int_0^{\alpha_2}\frac{1}{\eta}\int_0^{\alpha_2} X_{ij}^* d\alpha_2 d\alpha_2 - A_s^*\frac{1}{\eta}\int_0^{\alpha_2} X_{ij}^*\right)d\alpha_2 + \delta_{j0}U_i\int_0^h \rho d\alpha_2 +\right.\right.$$

$$\left.\left. + \delta_{j0}A_s^*\left(U_{ip} - U_i\right) + A_s^*\int_0^h S_{ij}\, d\alpha_2 - \int_0^h\left(\rho\int_0^{\alpha_2} S_{ij} d\alpha_2\right)\right] d\alpha_2\right\} + \left(\rho_h\frac{\partial h}{\partial t} - \frac{\partial}{\partial t}\int_0^h \rho d\alpha_2\right) +$$

$$+\left[\rho_0 U_2 + \rho_h\left(\frac{U_{1p}}{h_1}\frac{\partial h}{\partial \alpha_1} + \frac{U_{3p}}{h_3}\frac{\partial h}{\partial \alpha_3} - U_{2p}\right)\right]\delta_{j0}. \qquad (18)$$

$$A_S^* \equiv \int_0^h \rho A_S d\alpha_2, \quad A_\eta^* \equiv \int_0^h \rho A_\eta d\alpha_2, \qquad (19)$$

for j=0,1,2,… .

Mega Reynolds Equation (17) determines unknown pressure functions $p_j(\alpha_1,\alpha_2=0,\alpha_3)$ for j=0,1,2,…

5. Particular case

The following assumptions are made now:

1. Fluid viscosity η is independent of α_2 i.e. is constant in gap height direction. Then Eqs. (10), (19) give [6]:

$$A_s(\alpha_1,\alpha_3) \equiv \frac{\alpha_2}{h}, \quad A_s^*(\alpha_1,\alpha_3) \equiv \frac{\rho h}{2}, \quad A_\eta(\alpha_1,\alpha_3) \equiv \frac{\alpha_2}{2\eta}\left(\alpha_2 - h\right), \quad A_\eta^*(\alpha_1,\alpha_3) \equiv -\frac{\rho h^3}{12\eta}. \qquad (20)$$

2. Lubricant density ρ is constant.
3. We are neglecting the inertia forces of the lubricant i.e. $X_{ij}^* = 0$.
4. We take into account Newtonian fluid i.e. $S_{ij} = 0$.
5. Only one curvilinear surface is moving in α_1 direction, hence $U_1 \neq 0$, $U_2=U_3=0$, $U_{1p}=U_{2p}=U_{3p}=0$.
6. We have a stationary time independent flow.

Mega Reynolds Equation (17) for j=0 has the following form [6]:

$$\frac{\partial}{\partial\alpha_1}\left[\frac{h_3}{h_1}\frac{h^3}{\eta}\frac{\partial p_{d0}}{\partial\alpha_1}\right]+\frac{\partial}{\partial\alpha_3}\left[\frac{h_1}{h_3}\frac{h^3}{\eta}\frac{\partial p_{d0}}{\partial\alpha_3}\right]=6\frac{\partial}{\partial\alpha_1}(U_1h_3h), \tag{21}$$

for $h_1=h_1(\alpha_1,\alpha_3)$, $h_3=h_3(\alpha_1,\alpha_3)$.

6. Pressure changes in gap height direction

Now we are going to proof that the hydrodynamic pressure varies in gap height direction. From (3)-(5) for j=0 and conditions (8) we obtain velocity component and pressure in following form:

$$v_{10}(\alpha_1,\alpha_2,\alpha_3),\ v_{20}(\alpha_1,\alpha_2,\alpha_3),\ v_{30}(\alpha_1,\alpha_2,\alpha_3),\ p_{d0}(\alpha_1,\alpha_3). \tag{22.0}$$

Into Eqs. (3)-(5) for j=1 we put (22.0) and ζ=0 in (4). Hence, under conditions (8) we obtain corrections of velocity component and pressure in following form:

$$v_{11}(\alpha_1,\alpha_2,\alpha_3),\ v_{21}(\alpha_1,\alpha_2,\alpha_3),\ v_{31}(\alpha_1,\alpha_2,\alpha_3),\ p_{d1}(\alpha_1,\alpha_3). \tag{22.1}$$

……..

Into Eqs. (3)-(5) for j=J we put solutions (22.1), (22.2), ….(22.J-1) and ζ=0 in Eq.(4). Hence, under conditions (8) we obtain corrections of velocity component and pressure in following form:

$$v_{1J}(\alpha_1,\alpha_2,\alpha_3),\ v_{2J}(\alpha_1,\alpha_2,\alpha_3),\ v_{3J}(\alpha_1,\alpha_2,\alpha_3),\ p_{dJ}(\alpha_1,\alpha_3). \tag{22.J}$$

The above mentioned pressure functions we recognize as pressure and its corrections on the journal surface i.e. for α_2=0. Hence, we can write:

$$p_{d0}(\alpha_1,\alpha_3)=p_{d0}(\alpha_1,\alpha_2=0,\alpha_3),\ldots,p_{dJ}(\alpha_1,\alpha_3)=p_{dJ}(\alpha_1,\alpha_2=0,\alpha_3). \tag{23}$$

We multiply by D^J and sum up mutually Eq.(4) for j=0,1,…,J. Thus after integration both sides of modified Eq.(4) with respect to variable α_2 we obtain:

$$\sum_{j=0}^{J}D^j p_{dj}(\alpha_1,\alpha_2,\alpha_3)=\zeta\sum_{j=1}^{J}D^j F_j\left[v_{1j-1}(\alpha_1,\alpha_2,\alpha_3),v_{3j-1}(\alpha_1,\alpha_2,\alpha_3)\right]+C. \tag{24}$$

On the journal surface for α_2=0 the pressure is described by the formula:

$$\sum_{j=0}^{J}D^j p_{dj}(\alpha_1,\alpha_2=0,\alpha_3). \tag{25}$$

Hence the integration constant has the form:

$$\begin{aligned}C&=\sum_{j=0}^{J}D^j p_{dj}(\alpha_1,\alpha_2=0,\alpha_3)-\zeta\sum_{j=1}^{J}D^j\Phi_j(\alpha_1,\alpha_2=0,\alpha_3),\\ \Phi_j(\alpha_1,\alpha_2,\alpha_3)&\equiv F_j\left[v_{1j-1}(\alpha_1,\alpha_2,\alpha_3),v_{3j-1}(\alpha_1,\alpha_2,\alpha_3)\right].\end{aligned} \tag{26}$$

Pressure function has finally the form:

$$p_{d\Sigma}(\alpha_1,\alpha_2,\alpha_3)=\sum_{j=0}^{J}D^j p_{dj}(\alpha_1,\alpha_3)+\zeta\sum_{j=1}^{J}D^j\left[\Phi_j(\alpha_1,\alpha_2,\alpha_3)-\Phi_j(\alpha_1,\alpha_2=0,\alpha_3)\right]. \tag{27}$$

In particular case for J=1 we have:

$$p_{d1\Sigma}(\alpha_1,\alpha_2,\alpha_3)=\sum_{j=0}^{1}D^j p_{dj}(\alpha_1,\alpha_3)+\zeta D\left[\Phi_1(\alpha_1,\alpha_2,\alpha_3)-\Phi_1(\alpha_1,\alpha_2=0,\alpha_3)\right]=$$

$$=\sum_{j=0}^{1}D^j p_{dj}(\alpha_1,\alpha_3)+D\zeta\left[\left(\frac{\partial v_{10}}{\partial\alpha_2}\right)^2+\left(\frac{\partial v_{30}}{\partial\alpha_2}\right)^2\right]_{\alpha_2=0}^{\alpha_2}. \tag{28.1}$$

Velocity components v_{10}, v_{30} determined from (9) for j=0 we put in (28.1). Hence we obtain:

$$p_{d1\Sigma}(\alpha_1,\alpha_2,\alpha_3)=\sum_{j=0}^{1}D^j p_{dj}(\alpha_1,\alpha_3)+D\zeta\frac{\alpha_2}{\eta}\sum_{i=1,3}\frac{1}{h_i}\frac{\partial p_{d0}}{\partial\alpha_i}\left[\frac{\alpha_2-h}{\eta h_i}\frac{\partial p_{d0}}{\partial\alpha_i}-\frac{2U}{h}\delta_{i0}\right]. \quad (28.2)$$

On the sleeve surface for α_2=h the hydrodynamic pressure (28.2) has finally the form:

$$p_{d1\Sigma}(\alpha_1,\alpha_2=h,\alpha_3)=\sum_{j=0}^{1}D^j p_{dj}(\alpha_1,\alpha_3)-\frac{2DU}{\omega}\left(1+2\frac{\beta}{\alpha}\right)\frac{1}{h_1}\frac{\partial p_{d0}}{\partial\alpha_1}. \quad (29)$$

For cylindrical bearing $\alpha_1=\varphi$, α_2=r, α_3=z, $U=\omega R$, h_1=r. The pressure (29) on the sleeve has the form:

$$p_{d1\Sigma}(\varphi,r=h,z)=p_{d0}(\varphi,z)+Dp_{d1}(\varphi,z)-2D\left(1+2\frac{\beta}{\alpha}\right)\frac{\partial p_{d0}}{\partial\varphi}. \quad (30)$$

It is easy to see that if D=0, then pressure on the sleeve is identical as the pressure on the journal surface.

7. Adaptation of recurrences

In curvilinear coordinates $(\alpha_1,\alpha_2,\alpha_3)$ a modified Reynolds equations determines unknown function $p(\alpha_1,\alpha_3)$ {i.e. hydrodynamic pressure} in thin space between two surfaces with curvilinear sections. According to equations (17) an unknown function p for D=0 satisfies the following unified form of second order partial differential equation [10]:

$$C(\alpha_1,\alpha_3)\frac{\partial}{\partial\alpha_1}\left[A(\alpha_1,\alpha_3)\frac{\partial p}{\partial\alpha_1}\right]+F(\alpha_1,\alpha_3)\frac{\partial}{\partial\alpha_3}\left[B(\alpha_1,\alpha_3)\frac{\partial p}{\partial\alpha_3}\right]=A_\Sigma. \quad (31)$$

THEOREM
A partial homogeneous, second order differential Reynolds equation with variable continuous, single valued coefficients A, B, C, F, A_Σ derived in thin space between two movable surfaces in curvilinear, orthogonal coordinates (α_1, α_2, α_3) and presented in the following form:

$$C\frac{\partial A}{\partial\alpha_1}\frac{\partial p}{\partial\alpha_1}+CA\frac{\partial^2 p}{\partial\alpha_1^2}+F\frac{\partial B}{\partial\alpha_3}\frac{\partial p}{\partial\alpha_3}+FB\frac{\partial^2 p}{\partial\alpha_3^2}=A_\Sigma. \quad (32)$$

is simulated by the following linear, non-homogeneous partial second order recurrence equation with unknown function p, variable coefficients S and a variable free term Q:

$$S_\kappa(i,j)\,p_{i+1,j}+S_\nu(i,j)\,p_{i,j+1}+S_\pi(i,j)\,p_{i,j-1}+S_\xi(i,j)\,p_{i-1,j}-Z(i,j)\,p_{i,j}=Q(i,j). \quad (33)$$

7. Conclusions

An adaptation of the known recurrence and difference methods in the case of imposing of various boundary conditions that are formulated in curvilinear orthogonal coordinate systems during the solution of complex problems of Reynolds equations presents in the author's opinion a new scientific contribution, which is presented in this paper in the scope of linear recurrence equations with variable factors and a variable free term [5].

The mega-algorithms developed of the solutions and the properties of the mega-algorithm for the determination of the solutions of a generalized Reynolds partial recurrence equation with variable factors was used in numerical calculations with the use of professional software such as Matlab and Mathcad.

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