ANALYSIS OF DYNAMIC PROPERTIES OF HYDRAULIC LINES FOR FLUID POWER TRANSMISSION

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Abstract

This paper presents the results of a hydraulic line dynamic properties analysis taking into account inertia of the fluid flowing in rigid and flexible lines, the compressibility effect for this fluid and the viscous friction effect. The following are described and analyzed: solution of the wave equation in the form binding four variables: pressure and flow rate at the line input, and pressure and flow rate and the line output; two of the above-mentioned variables should be regarded as independent (input) and the other two as dependent (output), the accuracy comparison of distributed parameters model to lumped parameters model with regard to applicability range in hydraulic systems analysis, pressure value at given installation points as a response to rapid valve closing (transient response) or to valve opening (waterhammer effect). In these considerations the hydraulic line is regarded as a two-port with two inputs and two outputs with a definite transmittance matrix. Main considerations concern the variable resistance model. A general solution is given as a function of the Laplace operator. Introducing appropriate simplifications into the variable resistance model, the constant resistance model and lossless line model is obtained. Also, general solutions for three different lump parameter models are presented. For the lumped parameters line presents three equivalent models possible, i.e. as: a symmetrical two-port, a two-port with shared resistance and a two-port with capacitance at the output.

Keywords: *fluid power transmission, delivery of a pump, volume flow (rate), hydrostatic (forcing) pressure, the compressibility effect, the viscous friction effect*

1. Introduction

Contemporary fluid power transmission must feature high operating speed, high accuracy, and optimal energy consumption. To guarantee these features it is necessary to develop design methods accounting for fluid powering dynamical properties, and also during fluid powering analysis and synthesis.

The fluid power transmission is designed concerning the following:

- Hydraulic receiver (actuator or hydraulic actuator) with a control unit and its supply unit form a single, compact component,
- Hydraulic receiver is located at usually significant distance from its control and supply units (a long hydraulic line).

In the first case, a hydraulic line of fluid power transmission can be treated as stationary systems with lumped parameters, and physical phenomena taking place in this line can be described by means of a mathematical model, nonlinear or linearised, deterministic or probabilistic, depending on whether input and disturbance signals are immanently deterministic or stochastic. In the second case there is a need to consider the hydraulic line as a distributed parameter fluid power transmission.

Usually during the design routine the hydraulic line is regarded as a stationary with lumped parameters, and its dynamical properties are not taken into account in the computations. Dynamic properties of hydraulic line generate continuous or transient pulsations of circuit pressure and flow rates which cause machinery instabilities, fatigue of materials and other such harmful phenomena.

In the situation when a hydraulic receiver is located at some, often significant distance from its control and supply units, the hydraulic line should be regarded as a line in which the inertia of the fluid flowing in the line, compressibility effect for this fluid, and viscous friction effect have essential impact on the hydraulic system dynamics.

2. Hydraulic line with dispersed parameters as a circuit model

According to the assumptions made [4], hydraulic line properties are fully determined by the following PDE system in cylindrical coordinates [1, 2, 3]:

- fluid continuity equation (conservation of mass):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\rho r V_r + \frac{\partial}{\partial x}\rho V_x = 0\,, \tag{1}$$

where:

ρ - working fluid density,
r - hydraulic line running radius,
x - hydraulic line axial coordinates,
V_r, V_x - velocity components in the radial and axial direction, respectively,

- Navier-Stokes equation of motion in the direction of the flow (x axis):

$$\rho\left(\frac{\partial V_x}{\partial t} + V_r\frac{\partial V_x}{\partial r} + V_x\frac{\partial V_x}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial}{\partial x}\left[-\frac{2}{3r}\frac{\partial}{\partial r}\left(rV_r\right) + \frac{4}{3}\frac{\partial V_x}{\partial x}\right] + \frac{\mu}{r}\frac{\partial}{\partial r}r\left(\frac{\partial V_x}{\partial r} + \frac{\partial V_r}{\partial x}\right), \tag{2}$$

- equation of energy conservation (fluid state equation):

$$\frac{\partial \rho}{\rho} = \frac{\partial p}{B}\,, \tag{3}$$

where:

B - fluid compressibility modulus.

The equations from (1) to (3) are nonlinear. Taking into account that the lengths of pressure wave in the hydraulic line (guide) are substantially larger than the inner radius of the line in the frequency range present in hydraulic systems, these equations can be linearised.

After linearisation, the equation (1) for condition [1] $2\pi|c|/\omega \gg r_w$ takes the form:

$$\frac{\bar{\rho}}{B}\frac{\partial p'}{\partial t} + \bar{\rho}\frac{\partial V_x'}{\partial x} = 0\,, \tag{4}$$

where:

r_w - hydraulic line outer radius,
ω - wave frequency,
c - velocity of the wave propagating along the line,
$\rho, \bar{\rho}, \rho'$ - fluid density, its steady value and deviation from the steady value,
$V_x, \overline{V_x}, V_x'$ - axial velocity component, its steady value and deviation from the steady value,
$p, \bar{p}, p'$ - pressure, its steady value and deviation from the steady value.

After same transformations described in [4] the equation (2) takes the form:

$$\bar{\rho}\frac{\partial V_x'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x'}{\partial r}\right). \tag{5}$$

Applying the Laplace transform with respect to time with zero initial conditions $p(x,0) = 0$ and $V_x(x, r, 0) = 0$ and introducing a new variable:

$$U = V + \frac{1}{\bar{\rho} s}\frac{\partial P}{\partial x}, \tag{6}$$

the following equation is obtained:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{s}{\nu} U = 0. \tag{7}$$

It is a zero order modified Bessel equation. Its solution is a zero order Bessel function, and we consider only the Bessel function of the first kind which is finite at $r = 0$:

$$U = F(x, s)\left[J_0\left(jr\sqrt{\frac{s}{\nu}} \right) \right]. \tag{8}$$

After same transformations described in [4] one obtains differential equations of the hydraulic line with variable resistance:

$$Z(s)\, Q(x, s) = -\frac{\partial P(x, s)}{\partial x}, \tag{9}$$

$$Y(s)\, P(x, s) = -\frac{\partial Q(x, s)}{\partial x}, \tag{10}$$

where:

$$Z(s) = \frac{L_0 s}{1 - \dfrac{2}{j\sqrt{\dfrac{s}{\omega_0}}} \dfrac{J_1\left(j\sqrt{\dfrac{s}{\omega_0}} \right)}{J_0\left(j\sqrt{\dfrac{s}{\omega_0}} \right)}}, \tag{11}$$

$$Y(s) = C_0\, s, \tag{12}$$

and $L_0 = \dfrac{\bar{\rho}}{\pi r_w^2}$,

$$C_0 = \frac{\pi r_w^2}{a^2 \bar{\rho}} = \frac{\pi r_w^2}{B}.$$

Equations (9) and (10) can be written in the matrix form:

$$\frac{\partial}{\partial x}\begin{bmatrix} P(x, s) \\ Q(x, s) \end{bmatrix} = -\begin{bmatrix} 0 & Z(s) \\ Y(s) & 0 \end{bmatrix}\begin{bmatrix} P(x, s) \\ Q(x, s) \end{bmatrix}. \tag{13}$$

Expressing the hydraulic line equation in the matrix form enables its direct application for building a block diagram describing the hydraulic system dynamics and allows employment of the automatic control theory for analysis and synthesis.

By integrating the equation (13) with respect to x within the interval 0 to l and substituting (11) and (12) we obtain a solution for this equation at the complex plane:

$$\begin{bmatrix} P(0, s) \\ Q(0, s) \end{bmatrix} = \begin{bmatrix} ch\Gamma(s) & Z_c(s)\, sh\Gamma(s) \\ \dfrac{1}{Z(s)} sh\Gamma(s) & ch\Gamma(s) \end{bmatrix}\begin{bmatrix} P(l, s) \\ Q(l, s) \end{bmatrix}, \tag{14}$$

where:

$$\Gamma(s)=\frac{T_0 s}{\sqrt{1-\frac{2}{j\sqrt{\frac{s}{\omega_0}}}\frac{J_1\left(j\sqrt{\frac{s}{\omega_0}}\right)}{J_0\left(j\sqrt{\frac{s}{\omega_0}}\right)}}}, \tag{15}$$

is the propagation operator,

$$Z_c(s)=\frac{Z_{c0}}{\sqrt{1-\frac{2}{j\sqrt{\frac{s}{\omega_0}}}\frac{J_1\left(j\sqrt{\frac{s}{\omega_0}}\right)}{J_0\left(j\sqrt{\frac{s}{\omega_0}}\right)}}}, \tag{16}$$

is the wave impedance,
where:

$$T_0=l\sqrt{L_0C_0}=\frac{l}{a},\ Z_{c0}=\sqrt{\frac{L_0}{C_0}}=\frac{a\,\bar{\rho}}{\pi r_w^2}.$$

The equation (14) can be expressed in the form of the following two-port transmittance matrix:
- admittance form:

$$\begin{bmatrix}P(l,s)\\Q(0,s)\end{bmatrix}=\frac{1}{ch\Gamma(s)}\begin{bmatrix}1 & -Z_c(s)\,sh\Gamma(s)\\ \frac{1}{Z_c(s)}sh\Gamma(s) & 1\end{bmatrix}\begin{bmatrix}P(0,s)\\Q(l,s)\end{bmatrix}, \tag{17}$$

- impedance form:

$$\begin{bmatrix}P(0,s)\\P(l,s)\end{bmatrix}=\frac{Z_c(s)}{sh\Gamma(s)}\begin{bmatrix}ch\Gamma(s) & -1\\ 1 & -ch\Gamma(s)\end{bmatrix}\begin{bmatrix}Q(0,s)\\Q(l,s)\end{bmatrix}. \tag{18}$$

In the analysis of the model with distributed parameters, emphasis has been put on determination of the frequency response of the system. It is justified by the following:
- The already developed analysis and synthesis methods in the automatic control theory are based mainly on frequency characteristics,
- Frequency characteristics give full information on dynamical properties of the system,
- Computation of frequency characteristics in the case of transmittance with hyperbolic functions is less problematic than computation of transient responses.

Frequency characteristics are obtained by inserting $s = j\omega$ into the transmittance, or (equivalently) by exchanging the Laplace transform for the Fourier transform. Spectral forms of the propagation operator and of the wave impedance for the variable resistance model (15) and (16) are as follows:

$$\Gamma(j\omega)=\frac{jT_0\omega}{\sqrt{1-\frac{2}{j^{\frac{3}{2}}\sqrt{\frac{\omega}{\omega_0}}}\frac{J_1\left(j^{\frac{3}{2}}\sqrt{\frac{\omega}{\omega_0}}\right)}{J_0\left(j^{\frac{3}{2}}\sqrt{\frac{\omega}{\omega_0}}\right)}}}=\alpha+j\beta, \tag{19}$$

$$Z_c(j\omega) = \frac{Z_{c0}}{\sqrt{1 - \dfrac{2}{j^{\frac{3}{2}}\sqrt{\dfrac{\omega}{\omega_0}}} \dfrac{J_1\left(j^{\frac{3}{2}}\sqrt{\dfrac{\omega}{\omega_0}}\right)}{J_0\left(j^{\frac{3}{2}}\sqrt{\dfrac{\omega}{\omega_0}}\right)}}} = \gamma + j\delta \,, \tag{20}$$

where:

- real part of the propagation operator in the spectral form equals to:

$$\alpha = T_0\omega \frac{\sin\left(\dfrac{1}{2}\operatorname{arctg}\dfrac{b}{a}\right)}{\sqrt[4]{a^2+b^2}} \,,$$

- imaginary part of the propagation operator in the spectral form equals to:

$$\beta = T_0\omega \frac{\cos\left(\dfrac{1}{2}\operatorname{arctg}\dfrac{b}{a}\right)}{\sqrt[4]{a^2+b^2}} \,,$$

- real part of the wave impedance in the spectral form equals to:

$$\gamma = Z_{c0} \frac{\cos\left(\operatorname{arctg}\dfrac{b}{a}\right)}{\sqrt[4]{a^2+b^2}} \,,$$

- imaginary part of the wave impedance in the spectral form equals to:

$$\delta = -Z_{c0} \frac{\sin\left(\operatorname{arctg}\dfrac{b}{a}\right)}{\sqrt[4]{a^2+b^2}} \,.$$

Figure 1 and 2 present plots of real and imaginary parts of the propagation operator and the wave impedance as functions of frequency; these plots are given in dimensionless coordinates.

3. Hydraulic line with lumped parameters as a circuit model

In the description of the hydraulic line dynamical properties it is assumed that all system parameters are lumped, i.e.: resistance ($R_c = l\,R_0$), inertance ($L_c = l\,L_0$), capacitance ($C_c = l\,C_0$).

Equivalent diagram for the lumped parameters line as a symmetrical two-port is shown in Fig. 3. Equivalent diagram for the lumped parameters line as a two-port with shared resistance is depicted in Fig. 4. Equivalent diagram for the lumped parameters line as a two-port with capacitance at the output is shown in Fig. 5.

The line with lumped parameters as a symmetrical two-port, shown in Fig. 3, is described by the following equation in the matrix form:

$$\begin{bmatrix} P(0,s) \\ Q(0,s) \end{bmatrix} = \begin{bmatrix} \dfrac{C_cL_c}{2}s^2 + \dfrac{C_cL_c}{2}s + 1; & \dfrac{C_cL_c^2}{4}s^3 + \dfrac{C_cL_cR_c}{2}s^2 + \left(\dfrac{C_cR_c^2}{4} + L_c\right)s + R_c \\ C_cs; & \dfrac{C_cL_c}{2}s^2 + \dfrac{C_cL_c}{2}s + 1 \end{bmatrix} \begin{bmatrix} P(l,s) \\ Q(l,s) \end{bmatrix}. \tag{21}$$

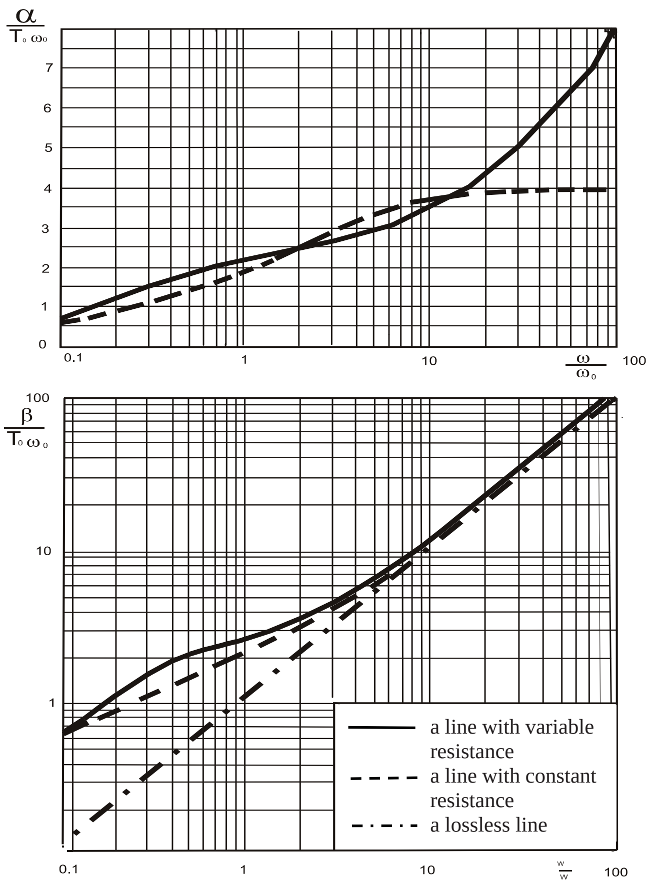

Fig. 1. Frequency dependence of real and imaginary parts of the wave propagation operator

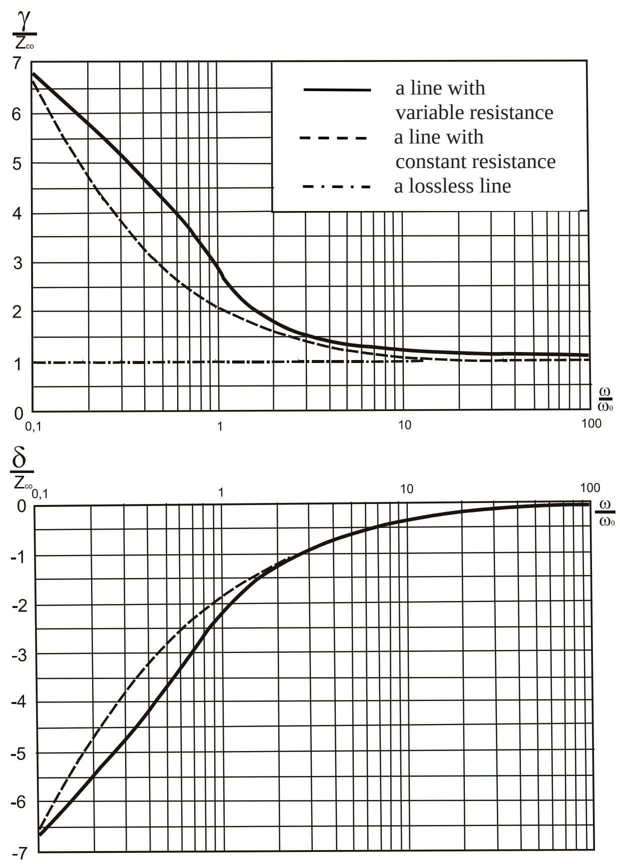

Fig. 2. Frequency dependence of real and imaginary parts of the wave impedance

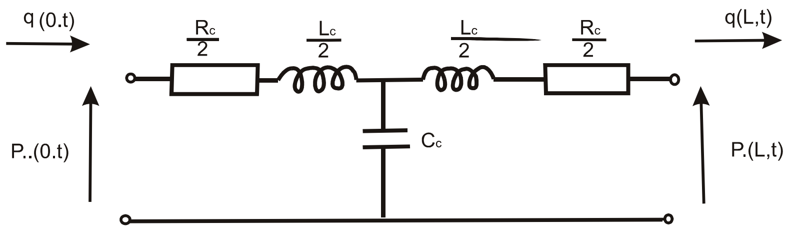

Fig. 3. Equivalent diagram for the lumped parameters line as a symmetrical two-port

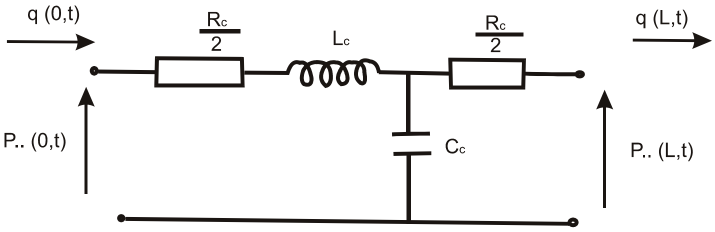

Fig. 4. Equivalent diagram for the lumped parameters line as a symmetrical two-port with shared resistance

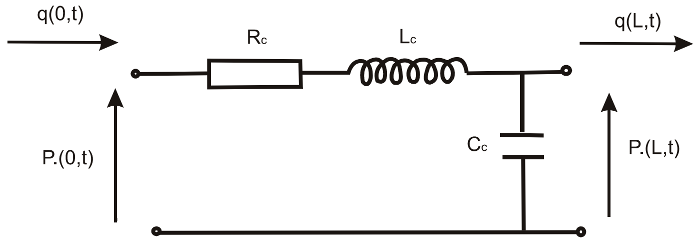

Fig. 5. Equivalent diagram for the lumped parameters line as a symmetrical two-port with capacitance at the output

The line with lumped parameters as a two-port with shared resistance, shown in Fig. 4, is described by the following equation in the matrix form:

$$\begin{bmatrix} P(0,s) \\ Q(0,s) \end{bmatrix} = \begin{bmatrix} C_c L_c s^2 + \frac{C_c R_c}{2} s + 1; & \frac{C_c L_c R_c}{2} s^2 + \left(\frac{C_c R_c^2}{4} + L_c \right) s + R_c \\ C_c s; & \frac{C_c R_c}{2} s + 1 \end{bmatrix} \begin{bmatrix} P(l,s) \\ Q(l,s) \end{bmatrix}. \tag{22}$$

The line with lumped parameters as a two-port with capacitance at the output, shown in Fig. 5, is described by the following equation in the matrix form:

$$\begin{bmatrix} P(0,s) \\ Q(0,s) \end{bmatrix} = \begin{bmatrix} C_c L_c s^2 + C_c R_c s + 1; & L_c s + R_c \\ C_c s; & 1 \end{bmatrix} \begin{bmatrix} P(l,s) \\ Q(l,s) \end{bmatrix}. \tag{23}$$

The models for the hydraulic line with lumped parameters given by the equations (21) and (23) are nonsymmetric.

Figure 6 shows the frequency characteristics for the dispersed parameters line models. These transmittances and the lumped parameters line transmittance are as follows:

- symmetrical two-port:

$$\frac{P(l,s)}{P(0,s)} = \frac{\dfrac{1}{1+K_z R_c}}{\dfrac{\frac{1}{4} K_z C_c L_c^2}{1+K_z R_c} s^3 + \dfrac{\frac{1}{2} C_c L_c + \frac{1}{2} K_z C_c L_c R_c}{1+K_z R_c} s^2 + \dfrac{\frac{1}{2} C_c R_c + \frac{1}{4} K_z C_c R_c^2 + K_z L_c}{1+K_z R_c} s + 1}, \tag{24}$$

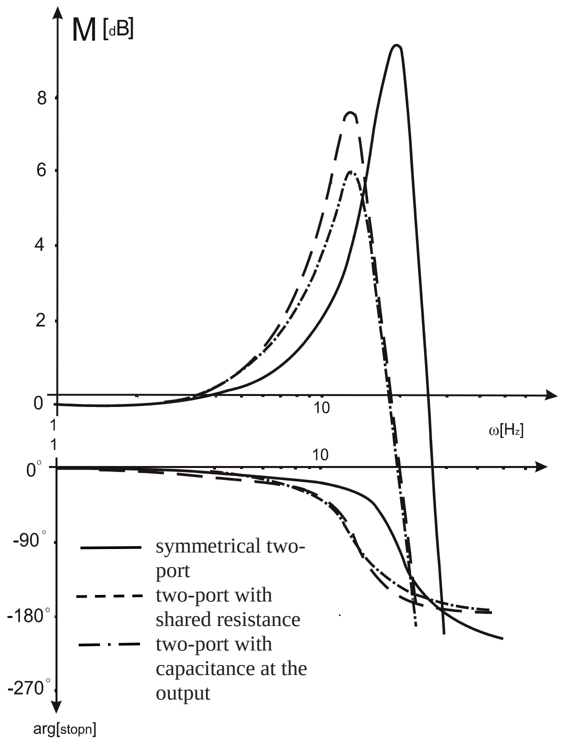

Fig. 6. Frequency characteristics for models with lumped parameters

- two-port with shared resistance:

$$\frac{P(l,s)}{P(0,s)} = \frac{\dfrac{1}{1+K_z R_c}}{\dfrac{C_c L_c + \frac{1}{2} K_z C_c L_c R_c}{1+K_z R_c} s^2 + \dfrac{\frac{1}{2} C_c R_c + \frac{1}{4} K_z C_c R_c^2 + K_z L_c}{1+K_z R_c} s + 1}, \tag{25}$$

- two-port with capacitance at the output:

$$\frac{P(l,s)}{P(0,s)} = \frac{\dfrac{1}{1+K_z R_c}}{\dfrac{C_c L_c}{1+K_z R_c} s^2 + \dfrac{C_c R_c + K_z L_c}{1+K_z R_c} s + 1}. \tag{26}$$

From Fig. 6 it follows that the most accurate model is the model of the line with lumped parameters treated as a symmetrical two-port.

4. Conclusions

The hydraulic line was characterised by means of the following hydraulic impedance (complex resistance) elements: series impedance consisting of an inertance and a resistance per unit length which accounts the inertia and viscous friction effects, and of a shunting admittance per unit length (characterised by capacitance) which accounts fluid compressibility effect. After integration of the wave equation with respect to the line length, the combination of the above parameters yielded two basic parameters characterizing models for the hydraulic line with dispersed parameters: the propagation operator and characteristic impedance. The first describes the time delay for signal transmission along the line, and damping and dispersion of pressure and flow rate waves. The second parameters are the line internal impedance observed from the point of view of the load.

Introducing appropriate simplifications into the variable resistance model, the constant resistance model and lossless line model was obtained. From the models with dispersed parameters, the computationally simplest one is the lossless line model. The lossless line model is recommended for analysis of transient processes (the so-called waterhammer effect) since it is easy to compute the inverse Laplace transform. The error arising due to negligence of dissipative losses is not so significant, because it moves the result to the safe side. It seems that models for the line with variable resistance reflect the hydraulic system properties in a most satisfactory way.

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