

DETERMINATION OF PSEUDO-VISCOSITY COEFFICIENTS FOR VISCO-ELASTIC LUBRICANTS

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Abstract

In this paper we assume determination of pseudo-viscosity coefficients of visco-elastic non-Newtonian fluid flow directly near the cooperating micro-bearing surface. We show the constitutive stress-strain dependencies of non-Newtonian visco-elastic media with regard to the content of addition of nano-particles.

The greater content of nano-particles and influences of superficial layer of cooperating surfaces on the dynamic oil viscosity, the more distinct non-Newtonian properties. The non-Newtonian properties create non-linear dependencies between strain and stress. Moreover the dynamic viscosity or apparent dynamic viscosity of oil often decreases along with shear rate increasing during motion. Dynamic viscosity of oil inside very thin boundary layers depends on Young's modulus of the cell body being in contact with the oil.

Considered visco-elastic oils with nano-particles have non-Newtonian pseudo-plastic properties, described in the beginning by means of Rivlin-Ericksen constitutive relations. Hence their stress-strain dependencies presented in this paper have the modified Rivlin-Ericksen form.

Scientific efforts brought into this elaboration represent an impact to the considered domain by providing for some data on comparisons between boundary conditions for lubrication flow in super thin boundary layer lying on the external surface of biological cells of joint cartilage and boundary conditions for lubrication of micro-bearing of shaft diameter smaller than 2 mm.

Keywords: pseudo elastic coefficients, method of determination

1. Preliminaries

In this paper we assume viscous and visco-elastic non-Newtonian oil flow directly near the cells of micro-bearing journal and sleeve surfaces. We show the constitutive stress-strain dependencies of non-Newtonian lubricants with regard to the content of addition of fluorocarbons and other various nano-substances [17]. The greater content of nano-substances and fluorocarbons inside the lubricant, the more distinct are the non-Newtonian properties.

The non-Newtonian properties create non-linear dependencies between strain and stress. Moreover the dynamic viscosity or apparent dynamic viscosity of non-Newtonian liquids with nano-particles and other additions often decreases along with shear rate increasing during motion.

Dynamic viscosity of non-Newtonian fluids inside very thin boundary layers depends on Young's modulus of the cell body being in contact. The Non-Newtonian ferro-lubricants and other fluids with fluorocarbons additions in super thin layers have pseudo-plastic and visco-elastic properties, caused by the small elastic additions of fluorocarbons and other elastic particles occurring during the micro-bearing lubrication.

2. Stress strain dependencies in Rivlin Ericksen model

Viscoelastic properties of biological fluids are described by means of Rivlin-Ericksen constitutive relations. Hence their stress-strain dependencies have the following form [3-7]:

$$\mathbf{S} = -p\mathbf{I} + \eta_0\mathbf{A}_1 + \alpha(\mathbf{A}_1)^2 + \beta\mathbf{A}_2 + \gamma(\mathbf{A}_1)^2\mathbf{A}_2, \quad (1a)$$

$$\mathbf{S} = -p\mathbf{I} + \eta_0\mathbf{A}_1 + \alpha(\mathbf{A}_1)^2 + \beta\mathbf{A}_2 + \delta\mathbf{A}_1\mathbf{A}_2, \quad (1b)$$

where: $\mathbf{A}_1, \mathbf{A}_2$ velocity deformation tensors.

Replacement of the tensors by their traces in terms describing visco-elastic biological properties of the fluid flowing in thin boundary layers has negligible low influence on the stresses. Hence the dependence (1a), (1b) we can be written in the following approximate form [4-7]:

$$\mathbf{S} \cong -p\mathbf{I} + \mathbf{A}_1 \left[\eta_0 + \alpha \operatorname{tr}\mathbf{A}_1 + \beta \frac{\operatorname{tr}\mathbf{A}_2}{\operatorname{tr}\mathbf{A}_1} + \gamma \operatorname{tr}\mathbf{A}_1 \operatorname{tr}\mathbf{A}_2 \right], \quad (2a)$$

$$\mathbf{S} \cong -p\mathbf{I} + \mathbf{A}_1 \left[\eta_0 + \alpha \operatorname{tr}\mathbf{A}_1 + \beta \frac{\operatorname{tr}\mathbf{A}_2}{\operatorname{tr}\mathbf{A}_1} + \delta \operatorname{tr}\mathbf{A}_2 \right], \quad (2a)$$

where the apparent viscosity has the following form [6, 7]:

$$\eta_p \equiv \eta_0 + \alpha \operatorname{tr}\mathbf{A}_1 + \beta \frac{\operatorname{tr}\mathbf{A}_2}{\operatorname{tr}\mathbf{A}_1} + \gamma \operatorname{tr}\mathbf{A}_1 \operatorname{tr}\mathbf{A}_2, \quad (3a)$$

$$\eta_p \equiv \eta_0 + \alpha \operatorname{tr}\mathbf{A}_1 + \beta \frac{\operatorname{tr}\mathbf{A}_2}{\operatorname{tr}\mathbf{A}_1} + \delta \operatorname{tr}\mathbf{A}_2, \quad (3b)$$

$$\mathbf{A}_1 \equiv \mathbf{L} + \mathbf{L}^T, \mathbf{A}_2 \equiv \operatorname{grad} \mathbf{a} + (\operatorname{grad} \mathbf{a})^T + 2\mathbf{L}^T\mathbf{L}, \quad (4)$$

$$\mathbf{a} \equiv \mathbf{L} \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}, \quad (5)$$

where:

\mathbf{A}_1 - tensor of deformation of the first kind [s^{-1}],

\mathbf{A}_2 - tensor of deformation of the second kind [s^{-2}],

$\operatorname{tr}\mathbf{A}_1$ - trace of tensor \mathbf{A}_1 ,

\mathbf{I} - dimensionless unit tensor,

\mathbf{L} - tensor of gradient of fluid velocity vector [s^{-1}],

\mathbf{L}^T - transpose tensor of gradient of fluid velocity vector [s^{-1}],

\mathbf{S} - stress tensor [Pa],

\mathbf{a} - acceleration vector [m/s^2],

p - pressure during the flow [Pa],

t - time [s],

\mathbf{v} - velocity vector [m/s],

α - first pseudo-viscosity experimental coefficient of the fluid [Pas^2],

β - second pseudo-viscosity coefficient of the fluid [Pas^2],

$\gamma \equiv \alpha^2\beta/\eta_0\eta_0$ - third experimental coefficient describing investigated fluid [Pas^4],

$\delta \equiv \alpha\beta/\eta_0$ - experimental coefficient describing investigated fluid [Pas^3],

η_0 - dynamic viscosity of motionless biological fluid or for the very slow movement of biological fluid [Pa·s],

η_∞ - dynamic viscosity of biological fluid in large motion [Pa·s],

η_p - apparent viscosity of biological liquid [Pa·s].

3. Experimental-theoretical methods

The classical stress –strain relation has the following form [1, 2]:

$$\mathbf{S} = -p\mathbf{I} + \eta_p\mathbf{A}_1, \quad (6)$$

where: η_p denotes apparent viscosity of the lubricant.

Majority of the experiments performed on the biological fluids indicate that dynamic viscosity

decreases along with shear rate increasing [1, 7]. Hence by virtue of the obtained experimental data and using the least square methods, we can express the viscosity -shear rate relation in the two following forms [1, 6, 7]:

$$\eta_p(A, B_a) = \eta_\infty + \frac{\eta_o - \eta_\infty}{1 + A \cdot \text{tr}(\mathbf{A}_1) + B_a \cdot [\text{tr}(\mathbf{A}_1)]^3 + B_a \cdot \text{tr}(\mathbf{A}_1)\text{tr}(\mathbf{A}_2)}, \quad (7a)$$

$$\eta_p(A, B_b) = \eta_\infty + \frac{\eta_o - \eta_\infty}{1 + A \cdot \text{tr}(\mathbf{A}_1) + B_b \cdot [\text{tr}(\mathbf{A}_1)]^2 + B_b \cdot \text{tr}(\mathbf{A}_2)}, \quad (7b)$$

where in both cases the coefficient A experimentally obtained, reaches values from 1.200s to 2.000s, and the coefficient B_a most often attains values from 0.00300 s² to 0.00600 s², B_b=0.00100s².

If we put A=0, B=0 then the relation (7a) or (7b) refers to the Newtonian classical liquid [6, 7]. We expand the apparent viscosity (7a) or (7b) being a function of two variables A and B_a, B_b, in Taylor series in the neighbourhood of the point A=0, B=0:

$$\eta_p(A, B_a) = \eta_p(A=0, B_a=0) + \frac{\partial \eta_p}{\partial A}(A=0, B_a=0)A + \frac{\partial \eta_p}{\partial B_a}(A=0, B_a=0)B_a + O(A^2) + O(B_a^2), \quad (8a)$$

$$\eta_p(A, B_b) = \eta_p(A=0, B_b=0) + \frac{\partial \eta_p}{\partial A}(A=0, B_b=0)A + \frac{\partial \eta_p}{\partial B_b}(A=0, B_b=0)B_b + O(A^2) + O(B_b^2). \quad (8b)$$

- In the case of dynamic viscosity determined by the equation (7a) we have:

$$\eta_p(A=0, B_a=0) = \eta_o, \quad (9)$$

$$\frac{\partial \eta_p}{\partial A}(A=0, B_a=0)A = \left. \frac{(\eta_o - \eta_\infty)(-\text{tr}\mathbf{A}_1)A}{[1 + A\text{tr}\mathbf{A}_1 + B_a(\text{tr}\mathbf{A}_1)^3 + B_a\text{tr}\mathbf{A}_1\text{tr}\mathbf{A}_2]^2} \right|_{A=0, B_a=0} = -(\eta_o - \eta_\infty)A \text{tr}\mathbf{A}_1, \quad (10)$$

$$\begin{aligned} \frac{\partial \eta_p}{\partial B_a}(A=0, B_a=0)B_a &= \left. \frac{-(\eta_o - \eta_\infty)[(\text{tr}\mathbf{A}_1)^3 + (\text{tr}\mathbf{A}_1)(\text{tr}\mathbf{A}_2)]B_a}{[1 + A\text{tr}\mathbf{A}_1 + B_a(\text{tr}\mathbf{A}_1)^3 + B_a\text{tr}\mathbf{A}_1\text{tr}\mathbf{A}_2]^2} \right|_{A=0, B_a=0} = \\ &= -(\eta_o - \eta_\infty)B_a [(\text{tr}\mathbf{A}_1)^3 - \text{tr}\mathbf{A}_1\text{tr}\mathbf{A}_2]. \end{aligned} \quad (11)$$

If we put mentioned derivatives into equation (8a) then we obtain:

$$\begin{aligned} \eta_p(A, B) &= \eta_o - (\eta_o - \eta_\infty)A \text{tr}(\mathbf{A}_1) - (\eta_o - \eta_\infty) \text{tr}(\mathbf{A}_1) \text{tr}(\mathbf{A}_1) \text{tr}(\mathbf{A}_1) B_a + \\ &- 1/2(\eta_o - \eta_\infty)B_a \text{tr}(\mathbf{A}_1)\text{tr}(\mathbf{A}_2) - 1/2(\eta_o - \eta_\infty)B_a \text{tr}(\mathbf{A}_1) \text{tr}(\mathbf{A}_2) + O(A^2) + O(B_a)^2. \end{aligned} \quad (13)$$

We equate both sides of equations (13), (3a). Thus we obtain following relations:

$$\alpha \text{tr}(\mathbf{A}_1) = -(\eta_o - \eta_\infty)[A + B_a(\text{tr}\mathbf{A}_1)^2] \text{tr}(\mathbf{A}_1), \quad (14)$$

$$\beta \text{tr}(\mathbf{A}_2) / \text{tr}(\mathbf{A}_1) = -1/2(\eta_o - \eta_\infty) B_a \text{tr}(\mathbf{A}_1)\text{tr}(\mathbf{A}_2), \quad (15)$$

$$\gamma \text{tr}(\mathbf{A}_1)\text{tr}(\mathbf{A}_2) \equiv \text{tr}(\mathbf{A}_1)\text{tr}(\mathbf{A}_2)\alpha^2\beta/(\eta_o)^2 = -1/2(\eta_o - \eta_\infty) B_a \text{tr}(\mathbf{A}_1)\text{tr}(\mathbf{A}_2). \quad (16)$$

We multiply both sides of the equation (15) by the quotient $\text{tr}\mathbf{A}_1/\text{tr}\mathbf{A}_2$ and both sides of the equation (14) we divide by the quotient $\text{tr}\mathbf{A}_1$. Then in this way obtained the equation (15) we put into the equation (14). On the basis of the introduced modifications we obtain:

$$\alpha - 2\beta = -(\eta_o - \eta_\infty)A. \quad (17)$$

Now we divide both sides of equation (16) by the factor $\text{tr}\mathbf{A}_1 \text{tr}\mathbf{A}_2$ and multiply by the $(\eta_0)^2$. Hence we obtain:

$$\alpha^2\beta = -\frac{1}{2}(\eta_0 - \eta_\infty)B_a(\eta_0)^2. \quad (18)$$

The set of equations (17), (18) leads to the following set of equations:

$$\alpha^3 + (\eta_0 - \eta_\infty)A\alpha^2 + (\eta_0 - \eta_\infty)\eta_0\eta_\infty B_a = 0, \quad \alpha - 2\beta = -(\eta_0 - \eta_\infty)A. \quad (19)$$

Set of equations (19) determines unknown pseudo-viscosity coefficients α , β as functions of known values A , B_a , η_0 , η_∞ .

- If dynamic viscosity is determined by the equation (7b), then we have:

$$\eta_p(A = 0, B_b = 0) = \eta_0, \quad (20)$$

$$\frac{\partial \eta_p}{\partial A}(A = 0, B_b = 0)A = \left. \frac{(\eta_0 - \eta_\infty)(-\text{tr}\mathbf{A}_1)A}{[1 + A\text{tr}\mathbf{A}_1 + B_b(\text{tr}\mathbf{A}_1)^2 + B_b\text{tr}\mathbf{A}_2]^2} \right|_{A=0, B_b=0} = -(\eta_0 - \eta_\infty)A \text{tr}\mathbf{A}_1, \quad (21)$$

$$\begin{aligned} \frac{\partial \eta_p}{\partial B_b}(A = 0, B_b = 0)B_b &= \left. \frac{-(\eta_0 - \eta_\infty)[(\text{tr}\mathbf{A}_1)^2 + (\text{tr}\mathbf{A}_2)]B_b}{[1 + A\text{tr}\mathbf{A}_1 + B_b(\text{tr}\mathbf{A}_1)^2 + B_b\text{tr}\mathbf{A}_2]^2} \right|_{A=0, B_b=0} = \\ &= -(\eta_0 - \eta_\infty)B_b [(\text{tr}\mathbf{A}_1)^2 + \text{tr}\mathbf{A}_2] \end{aligned} \quad (22)$$

If we put above derivatives into equation (8b) then we obtain:

$$\begin{aligned} \eta_p(A, B) &= \eta_0 - (\eta_0 - \eta_\infty)A \text{tr}(\mathbf{A}_1) - (\eta_0 - \eta_\infty) [\text{tr}(\mathbf{A}_1)]^2 B_b + \\ &- \frac{1}{2}(\eta_0 - \eta_\infty)B_b \text{tr}(\mathbf{A}_2) - \frac{1}{2}(\eta_0 - \eta_\infty)B_b \text{tr}(\mathbf{A}_2) + O(A^2) + O(B_b)^2. \end{aligned} \quad (23)$$

We equate both sides of equations (23), (3a). Thus we obtain following relations [7]:

$$\alpha \text{tr}(\mathbf{A}_1) = -(\eta_0 - \eta_\infty)[A + B_b \text{tr}\mathbf{A}_1] \text{tr}(\mathbf{A}_1), \quad (24)$$

$$\beta \text{tr}(\mathbf{A}_2) / \text{tr}(\mathbf{A}_1) = -\frac{1}{2}(\eta_0 - \eta_\infty) B_b \text{tr}(\mathbf{A}_2), \quad (25)$$

$$\gamma \text{tr}\mathbf{A}_2 \equiv \text{tr}\mathbf{A}_2 \alpha \beta / \eta_0 = -\frac{1}{2}(\eta_0 - \eta_\infty) B_b \text{tr}\mathbf{A}_2. \quad (26)$$

We divide both sides of the equation (25) by the quotient $\text{tr}\mathbf{A}_1 / \text{tr}\mathbf{A}_2$ and both sides of the equation (24) we divide by the factor $\text{tr}\mathbf{A}_1$. Then in this way obtained the equation (25) we put into the equation (24). On the basis of the introduced modifications we obtain:

$$\alpha - 2\beta = -(\eta_0 - \eta_\infty)A. \quad (27)$$

We divide both sides of equation (26) by the factor $\text{tr}\mathbf{A}_2$ and in such way obtained equation we multiply by the η_0 . After calculations we have:

$$\alpha\beta = -\frac{1}{2}(\eta_0 - \eta_\infty)\eta_0 B_b. \quad (28)$$

Solving the system of the algebraic equations (27), (28) we obtain the unknown values of material coefficients α and β for non-Newtonian oil as a function of known values A , B , η_0 , η_∞ , namely [7]:

$$\alpha = -\frac{1}{2}A(\eta_0 - \eta_\infty) - \frac{1}{2}\sqrt{[A(\eta_0 - \eta_\infty)]^2 - 4B_b\eta_0(\eta_0 - \eta_\infty)}, \quad (29)$$

$$\beta = \frac{1}{4}A(\eta_0 - \eta_\infty) - \frac{1}{4}\sqrt{[A(\eta_0 - \eta_\infty)]^2 - 4B_b\eta_0(\eta_0 - \eta_\infty)}, \quad (30)$$

In many cases we have $0 \ll (B/A)^2 \ll 1$ hence the terms multiplied by the factor $(B/A)^2$ are negligibly small. Then from the solutions (29), (30) the following approximate values yield:

$$\alpha \cong -(\eta_0 - \eta_\infty)A + \eta_0 B_b/A + O(B_b/A)^2, \quad (31)$$

$$\beta \cong \frac{1}{2} \eta_0 B_b/A + O(B_b/A)^2, \quad (32)$$

$$\gamma \cong -\frac{1}{2}(\eta_0 - \eta_\infty)B_b + O(B_b/A)^2. \quad (33)$$

The equations (31), (32), (33) show approximate unknown values material coefficients α , β , γ for pseudoplastic non-Newtonian oil as a functions of known values A , B_b , η_0 , η_∞ determined in experimental way.

4. Conclusions and numerical example

Performer considerations enable to determine the approximate values of material coefficients for pseudoplastic, non-Newtonian oil in the case if experimental determined parameters A , B and oil dynamic viscosity values η_0 , η_∞ where $\eta_0 > \eta_\infty$ for small and large shear rates are known.

The parameters A, B determine the decreases of the apparent viscosity of non-Newtonian oil if shear rate increases accordingly to the equations (7a), (7b).

Numerical example illustrates the usability of the obtained in his paper results.

Numerical example

The dynamic viscosity of the non-exploited oil has value $\eta_0 = 100 \text{ Pas}$ and the dynamic viscosity for the same oil but for large shear rates attain the value $\eta_\infty = 0.10 \text{ Pas}$. The parameters which determine the dynamic viscosity decreases for shear rate increases have the following values: $A = 1.88307 \text{ s}$, $B_b = 0.00458 \text{ s}^2$. By virtue of the equations (31), (32), (33) we have:

$$\alpha \cong -(100 \text{ Pas} - 0.10 \text{ Pas}) 1.88307 \text{ s} + 100 \text{ Pas} \cdot 0.00458 \text{ s}^2 / 1.88307 \text{ s}, \quad (34)$$

$$\beta \cong \frac{1}{2} 100 \text{ Pas} \cdot 0.00458 \text{ s}^2 / 1.88307 \text{ s}, \quad (35)$$

$$\gamma \cong -\frac{1}{2}(100 \text{ Pas} - 0.10 \text{ Pas}) 0.00458 \text{ s}^3. \quad (36)$$

After calculations we obtain: $\alpha = -187.8753 \text{ Pas}^2$, $\beta = 0.1216 \text{ Pas}^2$, $\gamma = -0.2287 \text{ Pas}^3$.

Considered oil after long time of exploitation had following values of dynamic viscosity and following parameters: $\eta_0 = 10.00 \text{ Pas}$, $\eta_\infty = 0.10 \text{ Pas}$, $A = 0.03349 \text{ s}$, $B_b = 0.00100 \text{ s}^2$. In this case from equations (31), (32), (33) follows: $\alpha = -0.03395 \text{ Pas}^2$, $\beta = 0.1478 \text{ Pas}^2$, $\gamma = -0.00495 \text{ Pas}^3$.

The values of coefficients α , β, γ are necessary to perform the right designing of micro-bearing lubricated with the non-Newtonian pseudo-plastic oils.

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