REDUCTION OF DEFORMATION IN A SPRING-MASS REALISATION OF HUMAN CHEST OCCURRED AFTER ACTION OF IMPACT

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Abstract

This paper takes into consideration active control of a mechanical construction modelling a human chest subject to impulsive elastic loading coming from a fast moving light mass. The effect of impact causes some essential deformation in the form of distance decreasing between the front and back side of the chest resulting from compression of internal organs. One can attenuate such a destructive process by introduction of fast-response active control elements attached to the front or back side of the chest i.e., from the direction of impacting mass or between back and support of the body.

The problem of one- or two-dimensional control is not easy to perform because of the consideration of the very short time of system's reaction. At this stage there has been used an effective numerical procedure for both solution and LQR control method application in a dynamical system of three separated elastically (rheologically as well) linked masses. Direction-dependent coefficients of the rheological link (extending our system to the seventh state-space dimension) govern different properties of internal organs during their stretching and compression. It puts into the control scheme's matrices time-dependent coefficients of damping influencing the optimal linear quadratic regulator used in control.

Keywords: *elastic impact, multi-dimensional dynamical system, rheological model, human chest, LQR algorithm, numerical simulation*

1. Introduction

In the area of widely developed new branch of biomechanics we can see the necessity of a deeper recognition of the character of constant or time-variable external loads influencing human organism. Some of more troublesome external loads for a human is a long-term or impulse impact interaction or its combinations. Among such phenomena the specific role is played by mechanical vibrations, or more precisely, their negative influence on the human organism working or existing in the conditions of harmful external surrounding. Unfavourable changes observed in human body are the consequences of e.g. professional exposition to vibrations with low or high frequencies and they depend on the place of penetration of these vibrations in the organism.

Mathematical methods of active vibration control are well developed [3] and they constitute the base for development of new techniques of human organism protection against undesirable vibration propagation. Optimal control methods [4], dynamic analysis and motion illustration in some human-machine couplings, which are to be developed in this article, can be adapted to improve conditions of life and work, to decrease side-effects arising during work or even during human daily life's activities. These methods of description and techniques of practical modelling of biomechanical interaction as well as the ways of negative influence compensation of external stimuli emitted to the human body vibrations of continuous or impulse character (e.g. chest protection against the effects of sudden impact of few kilos mass from the front on the level of breastbone) will be proposed. Mathematical analysis will be supported by quite new especially dedicated numerical procedures developed and optimized in the direction of solving multidimensional biomechanical elastic mass systems and also the systems of matrix equations resulting from the proposed schemes of active control and their modifications. Such techniques as optimization of quality index on the basis of the linear optimal regulator theory with a feedback and its effective modifications, robust control with the usage of singular perturbation approach and synthesis of optimal system of vibration isolation of the coupling: a sitting human – system of vibration's isolation constitute a starting point to the development of effective algorithms of active and passive compensation of vibrations. The scope of the proposed problems and provisional proposals of modelling and mathematical description is presented below.

Interesting and ambitious problem of dynamical description of biological organs is the active control of certain biomechanical system modelling a human chest subject to an elastic impact of a mass of light weight. The impact causes deformation of chest appearing in decrease of the distance between its front and back part and at the same time compressing internal organs. Such destructive action can be to a certain degree leveled by applying active element with a very short time of reaction attached to the front part of human chest from the side of impacting mass. The task of this element will be weakening of the impact force leading to decrease of the compressing stress amplitude. The problem of such control is not easy to realize that is why in the first approximation it will be accepted that human trunk propels from the back side (back part of chest) against the wall (continuation and more advanced numerical scheme with additional supporting reduction of deformation of interior chest with reaction force from the side of the back will constitute the broadening of this problem). In this way the possibility of chest's backward displacement as a result of impact will be restricted (such simplification in more advanced case will be omitted and some additional elastic stiffness and viscous damping will be introduced into the system). The problem will be mathematically discussed and the three-mass dynamical system with elastic coupling elements (also rheologic – see Fig. 1) written by means of a system of seven differential equations of first order will be numerically solved.

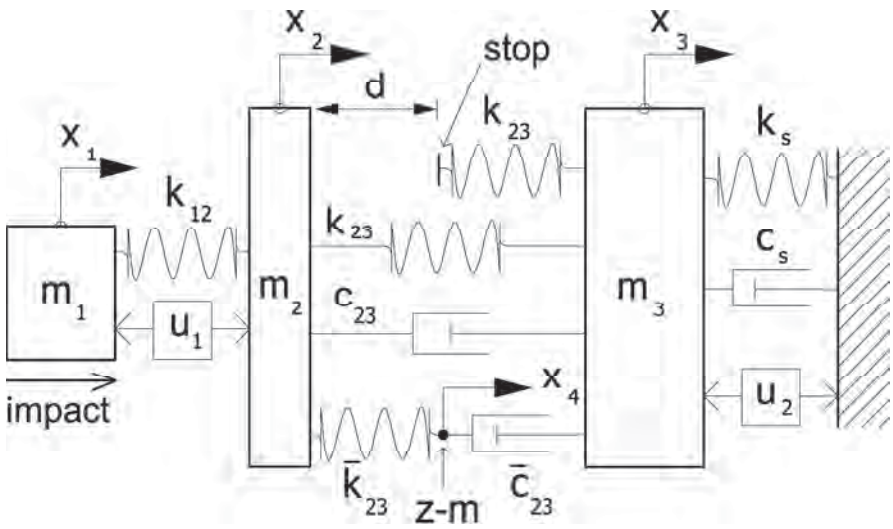

Fig. 1. Schematic 4DOF model of a chest's front m_2 and back m_3 sides and supported from behind with impulsively impacting mass m_1

The active control of the rheological structure of elastically connected masses concerns on some principles derived from a general approach [2]. Control of the investigated not subjected to any external loading three degrees of freedom parametrically discontinuous dynamical system represents a little particular case of the active control law used in the paper. There has been worked out a controlling scheme applied to the analyzed system, which after application of an initial velocity of impact coming from mass m_1 at initial time t_0 evaluates until the moment of time t_f is reached. The system is not influenced by any external disturbances affecting it from the surrounding environment.

2. Mathematical background of the problem

Let the following system of differential equations be given as follows:

$$\dot{\bar{x}}(t) = \mathbf{A}(t)\bar{x}(t) + \mathbf{B}\bar{u}(t), \qquad \bar{x}(t_0) = 0, \tag{1}$$

where:

A - (n×n) time-dependent matrix of structure parameters,
B - (n×n) time-constant matrix of attachment of executing (regulatory) elements,
x(t) - the n-dimensional state-space vector of the system,
u(t) - the p-dimensional vector of controlling forces.

Nomenclature convention used here and below uses an upper bar for notation of vectors in numbered formulas, bold italic font style for text-included notation of vectors and bold regular upper-case letters to denote matrices.

Our task focuses on searching for the control force ***u***(t) (see [2] for a particular description) that with some weighting matrices would satisfactorily minimize the cost function J in time $t = t_s$ for some time $t_s \in t_I = [t_0, t_f]$:

$$J = \frac{1}{2}\bar{x}(t_s)^T \bar{\theta}(t_s)\bar{x}(t_s) + \frac{1}{2}\int_{t_0}^{t_s} \begin{bmatrix} \bar{x}(t) \\ \bar{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{T} \\ \mathbf{T}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{u}(t) \end{bmatrix} dt, \tag{2}$$

where:

θ - time-dependent,
Q, **T**, **R** - constant weighting coefficient matrices.

2.1. Two-dimensional control force *u*(*t*)

Using a part of the theoretical background extended in [2] it is convenient to provide the following final relations leading to the general form of control law that will be identified and utilized in the next sections of the work. Following the above it is proposed that

$$\bar{u}(t) = -\mathbf{R}^{-1}\left(\mathbf{T}^T + \mathbf{B}^T\mathbf{K}_X\right)\bar{x}(t). \tag{3}$$

Equation (3) introduces one new matrix of (4n×4n) dimension called the Riccati matrix. Observe that the sought control law ***u***(t) is governed by a proportional relation to the systems' state vector. It is known and in our case necessary that the best method of estimation of the $\mathbf{K}_X$ matrix and thereby estimation of ***u***(t) is utilization of a proper convergent numerical procedure. This procedure solves the following matrix equation

$$\begin{aligned} &\left(\dot{\mathbf{K}}_X + \mathbf{K}_X\mathbf{A}_n + \mathbf{A}_n^T\mathbf{K}_X - \mathbf{K}_X\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_X + \mathbf{Q}_n\right)\bar{x}(t) = 0, \\ &\mathbf{A}_n = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{T}^T, \\ &\mathbf{Q}_n = \mathbf{Q} - \mathbf{T}\mathbf{R}^{-1}\mathbf{T}^T. \end{aligned} \tag{4}$$

A remark on the above, matrix $\mathbf{K}_X$ is symmetric along its trace so there will be obtained 4n first-order differential equations on its coefficients. These equations depend on all (sometimes time-varying) system parameters of stiffness k, dumping c and, in a consequence, coefficients of the introduced rheological properties of the interior of the chest and distance d describing our model. Therefore, the problem must be treated in a non-standard way. Because validity of Eq. (4) covers all ***x***(t)≠0, then the expression in braces preceding ***x***(t) must equal zero. Matrix $\mathbf{K}_X$ has to satisfy then the following Riccati matrix equation,

$$\dot{\mathbf{K}}_X + \mathbf{K}_X\mathbf{A}_n + \mathbf{A}_n^T\mathbf{K}_X - \mathbf{K}_X\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_X + \mathbf{Q}_n = 0, \tag{5}$$

also with inclusion of the final condition $\mathbf{K}_X(t_s) = \mathbf{K}_S$, where $\mathbf{K}_S$ is the matrix of known values selected at a time t_s (not necessarily at t_f) at which it would guarantee the minimal realization of J. Standard applications of such approach assume that values of $\mathbf{K}_S$ instantly converge to a set of constant values as $t \to t_f$. In the solution to our problem it is not assumed. Of course, equations in a number of $4n$ given in Eq. (5) must be numerically integrable in t_I. Thus, we are able to determine both the Riccati matrix $\mathbf{K}_S$ and the gain matrix $\boldsymbol{f}_x$ (for all $t \in t_I$),

$$\bar{f}_x(t) = -\mathbf{R}^{-1}\left(\mathbf{T}^T + \mathbf{B}^T\mathbf{K}_S\right). \tag{6}$$

Finally, the following control law can be proposed:

$$\bar{u}(t) = \bar{f}_x \bar{x}(t). \tag{7}$$

2.2. The analyzed dynamical system

Let $\mathbf{A}$ be a state matrix with some time-dependent (not constant) elements, and $\mathbf{B}$, $\mathbf{Q}$ and $\mathbf{R}$ some constant matrices. Matrix $\mathbf{K}_S$ of constant values only is a particular state of $\mathbf{K}_X$ at a time t_s. We have to regard to the standard approach that if the solution is stable for sufficiently large time t_f, the approximate values of $\mathbf{K}_X$ converge to an optimal matrix of $\boldsymbol{\theta}$. Evidently, one can assume that matrix $\mathbf{K}_S$ is constant and is in some state represented by the solution to the Riccati equation

$$\mathbf{K}_S\mathbf{A}_n + \mathbf{A}_n^T\mathbf{K}_S - \mathbf{K}_S\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_S + \mathbf{Q}_n = 0, \tag{8}$$

For the purpose of illustration of the route of the system's control law estimation the simple mechanical system visible in Fig. 1 of three degrees-of-freedom modelling the dynamics of rheological model of a human chest subject to a sudden loading from mass m_1 that is elastically connected with the front mass m_2 of the chest.

The next part of the application has to be preceded by some assumption. In order to control the system under investigation we have to additionally dispose of a virtual quick-response force generators of two control forces $\boldsymbol{u}(t) = [u_1(t), u_2(t)]^{\mathrm{T}}$ operating in two places means, between the first and second mass of the model (*direct impact force compensator*) and also between the chest's back mass m_3 and the not moveable wall (called here as a *back side force compensator*) as it is shown in Fig. 1. Equations of motion with an auxiliary force $\boldsymbol{u}(t)$ controlling the difference $(x_2 - x_3)$ are written in the following form

$$\begin{cases} \ddot{x}_1 = \dfrac{1}{m_1}\left[k_{12}(x_2 - x_1) - u_1\right], \\ \ddot{x}_2 = \dfrac{1}{m_2}\left[k_{12}(x_1 - x_2) - k_{23}(x_2 - x_3) - \bar{k}_{23}(x_2 - x_4) - c_{23}(\dot{x}_2 - \dot{x}_3) + u_1\right], \\ \ddot{x}_3 = \dfrac{1}{m_3}\left[k_{23}(x_2 - x_3) - k_s x_3 + c_{23}\dot{x}_2 - (c_{23} + \bar{c}_{23} + c_s)\dot{x}_3 + \bar{c}_{23}\dot{x}_4 - u_2\right], \\ \dot{x}_4 = \dot{x}_3 + \dfrac{\bar{k}_{23}}{\bar{c}_{23}}(x_2 - x_4), \end{cases} \tag{9}$$

and for the system ready for application of numerical solution we find it in a form of set of seven first-order differential equations given in Eq. (10).

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{x}_2 = v_2, \\ \dot{x}_3 = v_3 \\ \dot{x}_4 = v_3 + \dfrac{\bar{k}_{23}}{\bar{c}_{23}}(x_2 - x_4), \\ \dot{v}_1 = \dfrac{1}{m_1}\left[k_{12}(x_2 - x_1) - u_1\right], \\ \dot{v}_2 = \dfrac{1}{m_2}\left[k_{12}(x_1 - x_2) - k_{23}(x_2 - x_3) - \bar{k}_{23}(x_2 - x_4) - c_{23}(v_2 - v_3) + u_1\right], \\ \dot{v}_3 = \dfrac{1}{m_3}\left[k_{23}x_2 - (k_{23} + k_s)x_3 + c_{23}v_2 - (c_{23} + \bar{c}_{23} + c_s)v_3 + \bar{c}_{23}v_4 - u_2\right], \end{cases} \tag{10}$$

where v_4 in the last equation of system (10) is not the state-space variable and can be substituted by the fourth equation of (9) providing

$$\begin{cases} \vdots \\ \dot{v}_3 = \dfrac{1}{m_3}\left[(k_{23} + \bar{k}_{23})x_2 - (k_{23} + k_s)x_3 - \bar{k}_{23}x_4 + c_{23}v_2 - (c_{23} + c_s)v_3 - u_2\right] \end{cases}$$

State-space matrix $\mathbf{A}(t)$ and the controlling forces' matrix $\mathbf{B}$ defining their attachment points:

$$\mathbf{A}_{(7\times7)}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \dfrac{\bar{k}_{23}}{\bar{c}_{23}} & 0 & \dfrac{-\bar{k}_{23}}{\bar{c}_{23}} & 0 & 0 & 1 \\ \dfrac{-k_{12}}{m_1} & \dfrac{k_{12}}{m_1} & 0 & 0 & 0 & 0 & 0 \\ \dfrac{k_{12}}{m_2} & \dfrac{-k_{12} - k_{23}(t) - \bar{k}_{23}}{m_2} & \dfrac{k_{23}(t)}{m_2} & \dfrac{\bar{k}_{23}}{m_2} & 0 & \dfrac{-c_{23}(t)}{m_2} & \dfrac{c_{23}(t)}{m_2} \\ 0 & \dfrac{k_{23}(t) + \bar{k}_{23}}{m_3} & \dfrac{-k_{23}(t) - k_s}{m_3} & \dfrac{-\bar{k}_{23}}{m_3} & 0 & \dfrac{c_{23}(t)}{m_3} & \dfrac{-c_{23}(t) - c_s}{m_3} \end{bmatrix},$$

$$\mathbf{B}_{(2\times7)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{-1}{m_1} & \dfrac{1}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{-1}{m_3} \end{bmatrix}, \quad \bar{u}_{(2\times1)} = [u_1, u_2]^T.$$

Formally, some coefficients of $\mathbf{A}(t)$ are distance d-dependent (see Eq. (11)) but it implies its indirect dependency on time as the system can evaluate irregularly.

We propose a function of balance, which will precise the starting point as an input in the strategy of searching for the control force characteristics. Our fundamental task is to secure the amplitude of vibrations of the second mass minimized as low as possible in the preset interval of time $t \in (t_0, t_f)$. The system includes three time-dependent parameters that evaluate as below:

$$k_{23}(t) = \begin{cases} k_{23} & x_2 - x_3 < d, \\ 2k_{23} & x_2 - x_3 \geq d, \end{cases} \qquad \bar{c}_{23}(t) = \begin{cases} \bar{c}_{23} & v_2 - v_3 < 0, \\ 2\bar{c}_{23} & v_2 - v_3 \geq 0, \end{cases} \tag{11}$$

where d is a distance marked in Fig. 1. When controlled value $(x_2 - x_3)$ crosses over d then k_{23} doubles its value, as well as, when $(v_2 - v_3)$ changes its sign from negative to positive and vice versa, then $\overline{c_{23}}$ doubles its value too. In this way, there have been introduced basic rheological properties of the biomechanical model of our simply described human chest. Unquestionable drawback of that is confirmed because parameters of the system state-space matrix $\mathbf{A}$ and the Riccati matrix $\mathbf{K}_X$ too are time-dependent that will consequently require a special attention and application of some approach in determination of coefficients of the gain vector $\boldsymbol{f}_x$.

3. Control law and coefficients of gain matrix

The cost function (2) will be given in a more consisted form. Therefore, let **Q, R** and **T** satisfy

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & & & & & \\ & q_{22} & q_{23} & & & 0 & \\ & & q_{33} & q_{34} & & & \\ & & & q_{44} & q_{45} & & \\ & & & & q_{55} & q_{56} & \\ & 0 & & & & q_{66} & q_{67} \\ & & & & & & q_{77} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \quad \mathbf{T} = 0. \tag{12}$$

Matrices **Q** and **R** are responsible here for the quality and the reaction properties of the controlled system's response signal, respectively. Their coefficients are called weighting coefficients and, as it is seen, most of **Q** matrix's coefficients equal zero. Assumption of such a trace of q_{ij} coefficients of **Q** is arbitrary for this case and, for instance, with regards to a structure the dynamical system's equations q_{ij} for $i = 2\ldots7$, $j = 1\ldots6$ are zeroes. One should note that many of them do not improve the resulting response significantly, have minor influence or even being non-zero produce very irregular response. From other side, the bigger the amount of the coefficients is, the harder the response of the controlled system can be tuned.

Using constant matrices **Q** and **R** and applying assumptions listed above, the following form of cost function can be rewritten:

$$J = \frac{1}{2}\int_{t_0}^{t_s}\begin{bmatrix}\bar{x}(t)\\ \bar{u}(t)\end{bmatrix}^T\begin{bmatrix}\mathbf{Q} & 0\\ 0 & \mathbf{R}\end{bmatrix}\begin{bmatrix}\bar{x}(t)\\ \bar{u}(t)\end{bmatrix}dt,$$

$$J = \frac{1}{2}\int_{t_0}^{t_s}\left(x^T(t)\mathbf{Q}x(t) + u^T(t)\mathbf{R}u(t)\right)dt, \tag{13}$$

$$J = \frac{1}{2}\int_{t_0}^{t_s}\Big(v_3^2 q_{77} + v_2^2 q_{66} + v_1^2 q_{55} + x_4^2 q_{44} + x_3^2 q_{33} + x_2^2 q_{22} + x_1^2 q_{11} + x_3 x_4 q_{34} + x_2 x_3 q_{23} + \\ + x_1 x_2 q_{12} + u_1^2 r_{11} + u_2^2 r_{22} + u_1 u_2 (r_{12} + r_{21})\Big)dt.$$

We are interested in minimization of J in Eq. (13), and it is possible in this form (it was proved during numerical experiments that coefficients q_{45}, q_{56} and q_{67} do have a minor influence so they are disregarded) to find numerically in real as the trajectory of the controlled system evaluates in time. One can try to estimate the integral in Eq. (13) numerically for some values compositions of q_{ij} and r_{ij} in time $t \in [t_0, t_s]$ but there is 14 time-dependent square terms so it might be very difficult and cannot be applied directly, because it is also sensitive to coefficients of Riccati matrix and, it is more particularly taken into analysis in next section.

Estimation of $u(t)$ will be preceded by a numerical integration of Eq. (4) by means of the standard 4-*th* order Runge-Kutta procedure.

Components of Riccati matrix $\mathbf{K}_X$ during solution are denoted by

$$\forall_{i,j \in [1,7] \cup [1,7]} \mathbf{K}_X = [\alpha_{ij}], \quad \alpha_{ij} = \alpha_{ji}. \tag{14}$$

Substitution of the symmetric matrix $\mathbf{K}_X$ given by Eq. (14) to Eq. (4) produces 28 first order differential equations of which the first one takes the form:

$$\dot{\alpha}_{11} = \frac{r_{11}\alpha_{17}^2 m_1^2 m_2^2 + (r_{12} + r_{21})\alpha_{17} m_1 m_2 m_3 (\alpha_{16} m_1 - \alpha_{15} m_2)}{r' m_1^2 m_2^2 m_3^2} + \\ + \frac{m_3^2\left(r_{22}\alpha_{16}^2 m_1^2 - 2\alpha_{16} m_1 m_2 (r_{22}\alpha_{15} + r' k_{12} m_1) + m_2^2\left(r_{22}\alpha_{15}^2 + r' m_1 (2\alpha_{15} k_{12} - m_1 q_{11})\right)\right)}{r' m_1^2 m_2^2 m_3^2}, \tag{15}$$

$$r' = r_{11} r_{22} - r_{12} r_{21}.$$

In Eq. (15) α terms are time-dependent variables of 1-ODE and computed one by one during numerical integration of the whole set of the Riccati matrix's parameters. Next to them there are only constant terms of first stiffness of the system k_{12}, masses m_i, coefficients of reaction matrix **R** and q_{11}, the first coefficient of **Q** matrix. The problem of stable solution of the α_{ij} system could be not so difficult to investigate if all of ij (see Eq.(14)) first-order differential equations would contained only time-independent system parameters like k_{12}. There have been introduced in Eq.(11) non-constant system parameters (to describe the problem better in a bioengineering way) that will destabilize the solution for $\mathbf{K}_S$ coefficients denying for a convergence to a constant values, what would be desired for perfect realizations. One of such differential equations involving time-dependent system parameters takes the form:

$$\dot{\alpha}_{22} = \frac{2m_1^2 r'\left(\left(k_{23}(t)+\bar{k}_{23}\right)m_2^2 m_3 \alpha_{27} - \left(k_{12}k_{23}(t)\cdot\bar{k}_{23}\right)m_2 m_3^2 \alpha_{26}\right) + m_1^2 m_2^2 r_{11}\alpha_{27}^2}{r' m_1^2 m_2^2 m_3^2} + \\ + \frac{m_1 m_3\left(m_1\alpha_{26} - m_2\alpha_{25}\right)\left(\left(r_{12}+r_{21}\right)m_2\alpha_{27} + m_1 m_3 r_{22}\alpha_{26}\right)}{r' m_1^2 m_2^2 m_3^2} + \\ + \frac{m_2^2 m_3^2\left(r_{22}\alpha_{25}^2 - r'\left(2\alpha_{25}k_{12}m_1 + m_1^2\left(2\alpha_{24}\frac{\bar{k}_{23}}{\bar{c}_{23}} + q_{11}\right)\right)\right)}{r' m_1^2 m_2^2 m_3^2}. \tag{16}$$

Parameters $k_{23}(t)$ and $\bar{c}_{23}(t)$ are these unpredictable disturbance terms of the whole system of equations. It has a crucial consequence in the selection of the time at which one assumes that matrix $\mathbf{K}_S$ would guarantee proper coefficients of gain matrix $\boldsymbol{f}_x$ having the most important influence on control quality.

Equations for $\dot{\alpha}_{ij}$ (with its exemplary representatives shown in Eqs. (15) and (16)) are then integrated numerically since time t_s producing their final values:

$$\underset{i,j\in[1,7]\cup[1,7]}{\forall} \theta_{ij} = \lim_{t\to t_S}\mathbf{K}_S, \qquad \theta_{ij} = \theta_{ji}. \tag{17}$$

The knowledge of all components of $\mathbf{K}_S$ (with $\mathbf{T}$=0, see Eq. (6)) permits for calculation of the final time dependent gain matrix:

$$\bar{f}_x(t_s) = -\mathbf{R}^{-1}\mathbf{B}^T\bar{\theta} = \\ = \begin{bmatrix} \frac{-r_{22}}{r'} & \frac{r_{12}}{r'} \\ \frac{r_{21}}{r'} & \frac{-r_{11}}{r'} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{-1}{m_1} & 0 \\ \frac{1}{m_2} & 0 \\ 0 & \frac{-1}{m_3} \end{bmatrix}^T \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} & \theta_{16} & \theta_{17} \\ \theta_{12} & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} & \theta_{26} & \theta_{27} \\ \theta_{13} & \theta_{23} & \theta_{33} & \theta_{34} & \theta_{35} & \theta_{36} & \theta_{37} \\ \theta_{14} & \theta_{24} & \theta_{34} & \theta_{44} & \theta_{45} & \theta_{46} & \theta_{47} \\ \theta_{15} & \theta_{25} & \theta_{35} & \theta_{45} & \theta_{55} & \theta_{56} & \theta_{57} \\ \theta_{16} & \theta_{26} & \theta_{36} & \theta_{46} & \theta_{56} & \theta_{66} & \theta_{67} \\ \theta_{17} & \theta_{27} & \theta_{37} & \theta_{47} & \theta_{57} & \theta_{67} & \theta_{77} \end{bmatrix} = \\ = \begin{bmatrix} \frac{r_{12}\theta_{i7}}{r'm_3} + \frac{r_{22}}{r'}\left(\frac{\theta_{i6}}{m_2} - \frac{\theta_{i5}}{m_1}\right) & \cdots & \frac{r_{12}\theta_{67}}{r'm_3} + \frac{r_{22}}{r'}\left(\frac{\theta_{66}}{m_2} - \frac{\theta_{56}}{m_1}\right) & \frac{r_{12}\theta_{77}}{r'm_3} + \frac{r_{22}}{r'}\left(\frac{\theta_{67}}{m_2} - \frac{\theta_{57}}{m_1}\right) \\ -\frac{r_{11}\theta_{i7}}{r'm_3} - \frac{r_{21}}{r'}\left(\frac{\theta_{i6}}{m_2} - \frac{\theta_{i5}}{m_1}\right) & \cdots & -\frac{r_{11}\theta_{67}}{r'm_3} - \frac{r_{21}}{r'}\left(\frac{\theta_{66}}{m_2} - \frac{\theta_{56}}{m_1}\right) & -\frac{r_{11}\theta_{77}}{r'm_3} - \frac{r_{21}}{r'}\left(\frac{\theta_{67}}{m_2} - \frac{\theta_{57}}{m_1}\right) \end{bmatrix}, \\ i = 1\ldots5. \tag{18}$$

The resulting gain matrix is of dimension (2×7) and can be used in calculation of control forces accordingly to:

$$\bar{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \bar{f}_x\bar{x}(t) = \begin{bmatrix} f_{11}x_1 + f_{12}x_2 + f_{13}x_3 + f_{14}x_4 + f_{15}\dot{x}_1 + f_{16}\dot{x}_2 + f_{17}\dot{x}_3 \\ f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 + f_{25}\dot{x}_1 + f_{26}\dot{x}_2 + f_{27}\dot{x}_3 \end{bmatrix}, \tag{19}$$

where f_{ij} (i = 1,2, j = 1…7) are constant, but $\boldsymbol{x}(t)$ is the standard solution of non-controlled system. Procedure of numerical computations has to be executed subsequently from the standard solution of the dynamical system, through estimation of Riccati matrix's coefficients and coefficients of the gain matrix of the LQR algorithm finishing at the desired solution of controlled system. It is here necessary because of the non-stationary system taken into investigations.

4. Numerical simulation

The following set of system parameters is assumed: m_1=1.6, m_2=0.45, m_3=27 [kg], d=3.81[cm], k_{12} = 281, k_{23} = 26.3, $\bar{k}_{23}$ = 13.2 k_s = 10 [$\cdot 10^3$N/m], c_{23} = 1.23, $\bar{c}_{23}$ = 0.18, c_s = 0.11 [$\cdot 10^3$Ns/m] and initial conditions: $x_i(0) = 0$ (i = 1…4) [m], $v_1(0) = 13.9$, $v_i(0) = 0$ (i = 2…4) [m/s].

Figure 2 presents a measurable effect of application of the control scheme procedures (see [1] for some background) derived in previous sections. Difference between x_2 and x_3 has been taken as the measure of deformation of the chest's interior, so its only acceptable small values should secure internal organs from any undesirable compression and injuries.

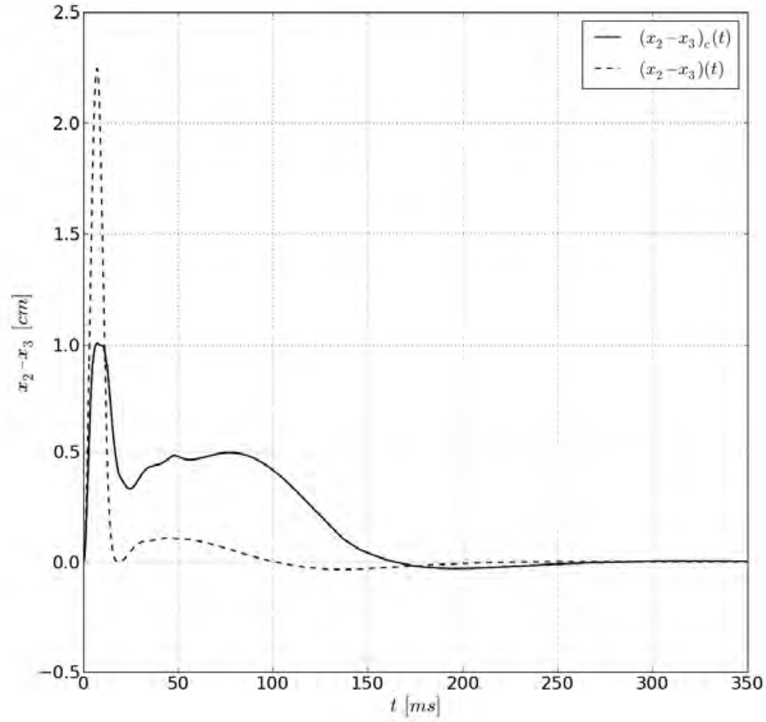

Fig. 2. Time histories of control variable (x_2-x_3) for standard (dashed line)and controlled (solid line) system solution

Control scheme coefficients q_{ij} and r_{ij} for the response characteristics visible in Fig. 2 are set as below (compare with Eq. (12)):

$$\begin{aligned} q_{i,i,(i=1\ldots7)} &= \left[-2.5\cdot10^5 \quad 1.15\cdot10^4 \quad -9\cdot10^4 \quad 9.5\cdot10^4 \quad 9.2\cdot10^{-1} \quad -1.85\cdot10^{-2} \quad -2.6\cdot10^2\right], \\ q_{i,i+1,(i=1\ldots6)} &= 10^4\cdot\left[0.2 \quad -1.0 \quad -1.0 \quad 0 \quad 0 \quad 0\right], \\ r_{ij,(i,j=1,2)} &= 10^{-3}\cdot\begin{bmatrix} 2.1 & 12.0 \\ -3.6 & 3.3 \end{bmatrix}. \end{aligned} \tag{20}$$

Set of constants of the control law has been found experimentally by execution of many attempts and comparison of the resulting graphs, so the final shape of the controlled system's response with application of two active elements as it was described may be not the most optimal. Nevertheless, it is now highly acceptable especially when maximal amplitudes of the two signals of the systems are compared. Maximum of the dashed-line time history reaches 2.246 [cm] while the solid-line time history of the controlled system response 1.006 [cm]. Thanks to the control scheme the difference amplitude $(x_2-x_3)_c$ has been reduced to $100\% \cdot \left(1 - \frac{2.246 - 1.006}{2.246}\right) \approx 44.8\%$ of its non-controlled original adequate (x_2-x_3).

5. Conclusions

The presented idea of active control of biomechanical structures is valid for the general concept of deformations minimization of some rheological elastic and continuous implementations of human body's models being under any external impulsive force action. The paper continues some previous investigations on possibilities of application of the performance index minimization method to fast time-changing models. For the purpose of solution of the biomechanical model given in the state-space representation with a time-dependent coefficients of the system state matrix it was necessary to develop some numerical methods allowing for the correct estimation of the control system coefficients. These were found and do not guarantee the best response characteristics of active element, but are quite good for the efficient reduction of the amplitude measuring the deformation of human internal organs. As it can be seen and what is worth mentioning here the model of description of the problem is relatively simple, solved by means of own dedicated and step by step created more universal numerical procedures (with a possibility to inspect of each command of the scheme) and can be proposed as an alternative to the complex finite element methods of commercial packages. Results are promising and confirm that the active control method is suitable to control fast response systems. One can try to imagine yourself such a fast reaction force real mechanism that could realize the desired type of dynamical response.

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